

THE EXISTENCE OF $r \times 4$ GRID-BLOCK DESIGNS WITH $r = 3, 4$ *

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Abstract. For a v -set V , let \mathcal{A} be a collection of $r \times c$ arrays with elements in V . A pair (V, \mathcal{A}) is called an $r \times c$ grid-block design if every two distinct elements i and j in V occur exactly once in the same row or in the same column of an array in \mathcal{A} . This design originated from the use of DNA library screening. In this paper, we show the existence of $r \times 4$ grid-block designs with $r = 3, 4$. We settle completely for the case of $r = 4$ and almost completely for the case of $r = 3$, leaving 15 orders undetermined.

Key words. grid-block design, complete graph, decomposition, Cartesian product

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1. Introduction and preliminaries. Let V be a v -set and \mathcal{A} be a collection of $r \times c$ arrays with elements in V . Two elements of V are *collinear* if they are on the same grid line (row or column). A pair (V, \mathcal{A}) is called an $r \times c$ grid-block design of order v if every pair of two distinct elements of V is collinear exactly once. Therefore, an $r \times c$ grid-block design of order v exists if and only if the complete graph of order v , K_v , can be decomposed into the Cartesian product of two complete graphs K_r and K_c , denoted by $G(r, c) = K_r \times K_c$. For graph terms not defined here, we refer the reader to [13].

For convenience, we denote an $r \times c$ grid-block design of order v by a $D_{r \times c}(K_v)$. It is not difficult to conclude the following necessary conditions for the existence of a $D_{r \times c}(K_v)$.

LEMMA 1.1. *If a $D_{r \times c}(K_v)$ exists, then (a) $v - 1 \equiv 0 \pmod{r + c - 2}$ and (b) $v(v - 1) \equiv 0 \pmod{rc(r + c - 2)}$.*

The study of the existence of a $D_{r \times c}(K_v)$ dates back to around 70 years ago when the case $r = c = v$ was first considered by Yates [15]. Such a grid-block is known as a lattice square. Later, in 1971, Raghavarao [12] gave a construction of a $D_{r \times c}(K_v)$ for the case when \sqrt{v} is an odd prime. In 1995, it was pointed out by Hwang [8] that this design can be applied to DNA clone library screening. For practical use, in a 2-stage group test, we need an $r \times c$ grid-block design of order v with smaller r and c

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comparable to v ; see [11, 10] for reference.

The existence of a $D_{2 \times 3}(K_v)$ for all admissible v was shown by Carter [4] in her Ph.D. thesis. There are three other substantial works on this topic. First, in 2003, Mutoh et al. [11] proved that the necessary conditions for the existence of a $D_{2 \times 4}(K_v)$ are also sufficient. Then, in 2004, Fu et al. [6] showed the existence of a $D_{3 \times 3}(K_v)$. Finally, in 2007, Li et al. [9] gave a complete solution to the existence of a $D_{2 \times 5}(K_v)$. In this paper, we shall extend their study and consider the grid-blocks $G(3, 4)$ and $G(4, 4)$.

The methods which we adopt are mainly direct constructions. For smaller grid-block designs (ingredients), the constructions are based on the familiar difference method, where a finite group (normally an Abelian group but sometimes a non-Abelian group) will be utilized to generate the set of grid-blocks for a given design. Thus, instead of listing all the grid-blocks, we list a set of base grid-blocks and generate the others by an additive group and perhaps some further automorphisms. For example, the following $G(4, 4)$ generates all 97 grid-blocks of a $D_{4 \times 4}(K_{97})$; the i th block is obtained by adding $i \pmod{97}$ to each element of this block.

0	1	3	7
5	13	81	38
16	60	26	86
46	74	61	29

In order to combine the ingredients together nicely to obtain a larger one, we need some notation on combinatorial designs. For terminology in design theory, we refer the reader to [5]. For sets of positive integers K and M , let V be a set of v elements, let \mathcal{g} be a partition of V such that each $G \in \mathcal{g}$ has m elements for $m \in M$, and let \mathbf{B} be a collection of k -subsets (blocks) of V for $k \in K$. A triple $(V, \mathcal{g}, \mathbf{B})$ is called a *group divisible design* (GDD), denoted by $GD[K, \lambda, M; v]$, if every two distinct elements contained in different groups of \mathcal{g} occur in exactly λ blocks of \mathbf{B} and every two distinct elements contained in the same group occur in no blocks. In particular, a $GD[\{k\}, \lambda, \{m\}; v]$ is written by $GD[k, \lambda, m; v]$ and called a k -GDD for simplicity of notation. Furthermore, if \mathcal{g} contains t_i groups of size s_i for $i = 1, 2, \dots, h$, then the k -GDD is said to be of type $\prod_{i=1}^h s_i^{t_i}$. Now, if $\lambda = 1$, then the existence of a $GD[k, 1, m; v]$ is equivalent to the decomposition of $K_n(m)$ into K_k 's, where $v = mn$ and $K_n(m)$ denotes the balanced complete n -partite graph with each partite set of size m . For convenience, the elements in each partite set will be called the points throughout this paper. A *pairwise balanced design* (PBD) with set of block sizes K is a K -uniform set system (X, \mathcal{A}) , such that every element of $\binom{X}{2}$ is contained in exactly one block of \mathcal{A} . A PBD of order v and set of block sizes K is denoted by $(v, K, 1)$ -PBD. If an element $k \in K$ is "starred" (written k^*), it means that the PBD has exactly one block of size k . The following lemma is easy to see.

LEMMA 1.2. (a) $(n-1)m \equiv 0 \pmod{r+c-2}$ and (b) $nm^2(n-1) \equiv 0 \pmod{rc(r+c-2)}$ are necessary conditions for the existence of a $D_{r \times c}(K_n(m))$.

Note here that we use a $D_{r \times c}(G)$ to denote a decomposition of G into $G(r, c)$'s.

The following recursive constructions are essential to the constructions of our main results. These constructions were obtained earlier by Fu et al. [6] and Mutoh et al. [11].

LEMMA 1.3 (see [6]). A $D_{r \times c}(K_{st})$ exists if a $D_{r \times c}(K_t)$ and a $D_{r \times c}(K_s(t))$ exist.

LEMMA 1.4 (see [6]). A $D_{r \times c}(K_{st+1})$ exists if a $D_{r \times c}(K_{t+1})$ and a $D_{r \times c}(K_s(t))$ exist.

LEMMA 1.5 (see [11]). *There exists a $D_{r \times c}(K_{vt+1})$ if there exist a $GD[K, 1, M; v]$, a $D_{r \times c}(K_{mt+1})$, and a $D_{r \times c}(K_k(t))$ for any $m \in M$ and $k \in K$.*

LEMMA 1.6 (see [11]). *There exists a $D_{r \times c}(K_s(tu))$ if there exist a $D_{r \times c}(K_s(t))$ and an $OA(u, k)$, where $k = \max\{r, c\}$ and $OA(u, k)$ is an orthogonal array of order u , degree k , and index 1.*

We shall also use the idea of Wilson in [14] to obtain grid-block designs of prime power orders. For a positive integer e , let q be a prime power such that $q \equiv 1 \pmod{e}$. Then, the cyclic multiplicative subgroup of nonzero elements in the finite field of order q , $GF(q)$, has a unique subgroup C_0^e of index e . The multiplicative cosets of $C_0^e : C_0^e, C_1^e, \dots, C_{e-1}^e$, are called the classes of index e . Now, let $A = (a_{ij})$ be an $r \times c$ grid-block with elements in $GF(q)$. Then, we define the ordered list of differences of A as $\vec{\partial}A = \{a_{jl} - a_{il} | 1 \leq i < j \leq r, 1 \leq l \leq c\} \cup \{a_{lj} - a_{li} | 1 \leq l \leq r, 1 \leq i < j \leq c\}$.

By a similar idea as the one in [8], we have the following result.

LEMMA 1.7. *For a prime power $q \equiv 1 \pmod{rc(r+c-2)}$, let $e = rc(r+c-2)/2$. If there exists an $r \times c$ array $A = (a_{ij})$ over $GF(q)$ such that $\vec{\partial}A$ is a system of representatives for the cosets of C_0^e , then a $D_{r \times c}(K_q)$ exists.*

We remark here that A can be viewed as a base grid-block and all the grid-blocks of $D_{r \times c}(K_q)$ can be obtained by calculating $cA+x$ for $c \in C_0^e/\{1, -1\}$ and $x \in GF(q)$. Now, we are ready to state the main results.

2. 3 × 4 grid-block designs. By Lemma 1.1, it is a routine matter to show the following fact.

LEMMA 2.1. *If a $D_{3 \times 4}(K_v)$ exists, then $v \equiv 1, 16, 21, 36 \pmod{60}$.*

For convenience, the constructions are split into four parts. (We claim that the above necessary conditions are sufficient except for $v = 16$, which is impossible, as well as several unsettled cases.)

2.1. $v \equiv 1 \pmod{60}$. The following lemmas provide the constructions of the ingredients.

LEMMA 2.2. *There exist both a $D_{3 \times 4}(K_4(20))$ and a $D_{3 \times 4}(K_5(15))$. Hence, there exist both a $D_{3 \times 4}(K_4(60))$ and a $D_{3 \times 4}(K_5(60))$.*

Proof. A $D_{3 \times 4}(K_4(20))$ is constructed on the Abelian group $V = Z_{80}$ which has only one base grid-block:

0	1	3	10
11	16	54	33
66	47	21	60

A $D_{3 \times 4}(K_5(15))$ is constructed on the Abelian group $V = Z_{75}$ generated by the group V acting on the points. This design has one base grid-block:

0	1	3	7
14	38	66	25
43	60	24	51

Since there are both an $OA(3, 4)$ and an $OA(4, 5)$, a $D_{3 \times 4}(K_4(60))$ and a $D_{3 \times 4}(K_5(60))$ can be obtained by applying Lemma 1.6. \square

LEMMA 2.3. *There exists a $D_{3 \times 4}(K_{60t+1})$ for $t = 1, 2, 3, 7$.*

Proof. According to Lemma 1.7, we use a computer to find the array A for $t = 1, 2, 3, 7$. These A 's are listed in the following table.

t	Primitive element or polynomial	A			
1	2	0	1	3	7
		5	25	56	43
		19	47	30	59
2	$\omega^2 + \omega^1 + 7$	ω^∞	ω^0	ω^1	ω^2
		ω^3	ω^5	ω^{15}	ω^{98}
		ω^{24}	ω^{95}	ω^{97}	ω^{45}
3	2	0	1	3	7
		5	13	23	63
		99	90	142	39
7	2	0	1	3	7
		5	13	23	37
		105	365	281	86

Now we can show the existence of the desired grid-block designs. Here, we verify the case $t = 1$, and the others are similar. Clearly, $q = 61$, $e = 30$, and $f = (q - 1)/e = 2$. We denote $F^* = GF(q) \setminus \{0\}$, ω to be the primitive element of F^* and $\varepsilon = \omega^e$. Then the cosets of C_0^e are as follows:

$C_0^e = \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{f-1}\} = \{1, 60\}$
$C_1^e = \{\omega, \omega\varepsilon, \omega\varepsilon^2, \dots, \omega\varepsilon^{f-1}\} = \{2, 59\}$
$C_2^e = \{\omega^2, \omega^2\varepsilon, \omega^2\varepsilon^2, \dots, \omega^2\varepsilon^{f-1}\} = \{4, 57\}$
\vdots
$C_{e-2}^e = \{\omega^{e-2}, \omega^{e-2}\varepsilon, \omega^{e-2}\varepsilon^2, \dots, \omega^{e-2}\varepsilon^{f-1}\} = \{15, 46\}$
$C_{e-1}^e = \{\omega^{e-1}, \omega^{e-1}\varepsilon, \omega^{e-1}\varepsilon^2, \dots, \omega^{e-1}\varepsilon^{f-1}\} = \{30, 31\}$

Finally, we compute $\vec{\partial}A$ and take all the 30 differences less than 31; that is, if the difference is 41, we write it down as 20. So the differences are $\vec{\partial}A = (5, 14, 19, 24, 15, 22, 8, 26, 27, 25, 16, 9, 1, 2, 3, 4, 6, 7, 20, 30, 10, 13, 18, 23, 28, 17, 11, 29, 12, 21)$. The 30 different differences correspond exactly to the elements 1 to 30. According to the cosets of C_0^e , we know that the elements 1 to 30 are exactly in the 30 different cosets. So, by Lemma 1.7, a $D_{3 \times 4}(K_{61})$ exists. Note that all the grid-blocks are obtained by calculating $cA + x$ for $c \in C_0^e/\{1, -1\}$ and $x \in GF(q)$. So, in fact, we have 61 grid-blocks in this case. \square

The following lemmas show that a $D_{3 \times 4}(K_{60t+1})$ exists for $t = 4, 5, 6, 8, 9, 10, 11$.

LEMMA 2.4. *There exist both a $D_{3 \times 4}(K_{241})$ and a $D_{3 \times 4}(K_{301})$.*

Proof. By Lemmas 2.2 and 2.3, there exist both a $D_{3 \times 4}(K_4(60))$ and a $D_{3 \times 4}(K_{61})$. Thus, there exists a $D_{3 \times 4}(K_{241})$ by Lemma 1.4. Again, by Lemmas 2.2 and 2.3, there exist both a $D_{3 \times 4}(K_5(60))$ and a $D_{3 \times 4}(K_{61})$. Hence, we have a $D_{3 \times 4}(K_{301})$. \square

LEMMA 2.5. *There exists a $D_{3 \times 4}(K_{361})$.*

Proof. First, we construct a $D_{3 \times 4}(K_6(60))$. We know a 5-GDD of type 4^6 exists; see [7] for a reference. Then we inflate each point in each partite set to 15 points. In other words, we view each point in each partite set as a subset (each of size 15). So, we can get a $D_{3 \times 4}(K_6(60))$ by using Lemma 2.2 to obtain 75 grid-blocks for each K_5 in a K_5 -design of $K_6(4)$. Since a $D_{3 \times 4}(K_{61})$ exists (Lemma 2.3), we get a $D_{3 \times 4}(K_{60 \times 6 + 1})$ by Lemma 1.4. \square

LEMMA 2.6. *There exists a $D_{3 \times 4}(K_8(15))$.*

Proof. Let the last partite set of $K_8(15)$ be $\{\infty_0, \infty_1, \infty_2, \dots, \infty_{13}, \infty_{14}\}$ and $V = Z_{105} \cup \{\infty_0, \infty_1, \infty_2, \dots, \infty_{13}, \infty_{14}\}$. Then, the design is constructed on the

group V generated by the group Z_{105} acting on the points. The 15 infinite points remain fixed under the action of Z_{105} . It has two base grid-blocks:

0	1	3	9	0	48	23	68
4	14	19	31	75	86	∞^*	34
40	87	58	∞	31	5	77	∞^s

We remark here that each of the above grid-blocks has an extra property that all elements in the same row or same column with ∞ (∞^* or ∞^s) are in different residue classes (mod 5). Therefore, for the 21 grid-blocks generated by adding $i, 5+i, \dots, 100+i$, $i = 0, 1, 2, 3, 4$, to every point in each base grid-block, respectively, we replace three infinite points ($\infty, \infty^*, \infty^s$) with $\infty_i, \infty_{i+5}, \infty_{i+10}$ correspondingly. Then, we have the desired $D_{3 \times 4}(K_8(15))$. \square

LEMMA 2.7. *There exists a $D_{3 \times 4}(K_{481})$.*

Proof. By Lemma 2.6, we have a $D_{3 \times 4}(K_8(15))$. Then we can get a $D_{3 \times 4}(K_8(60))$ by applying Lemma 1.6 with an $OA(4,4)$. Since a $D_{3 \times 4}(K_{61})$ exists, the desired grid-block design can be obtained by Lemma 1.4. \square

LEMMA 2.8. *There exists a $D_{3 \times 4}(K_9(15))$.*

Proof. The design is constructed on the Abelian group $V = Z_{135}$ generated by the group V acting on the points. This design has two base grid-blocks:

0	1	3	7	0	14	61	83	\square
5	13	24	37	50	115	89	66	
15	44	79	104	98	58	5	25	

LEMMA 2.9. *There exists a $D_{3 \times 4}(K_{541})$.*

Proof. By the fact that a $(37, \{5, 9^*\})$ -PBD exists (see [2] for a reference), we can get a $\{5, 9\}$ -GDD of type 4^9 by deleting one point not in the block of size 9. Then, by a similar idea as in Lemma 2.5, the existence of a $D_{3 \times 4}(K_5(15))$ and a $D_{3 \times 4}(K_9(15))$ gives the $D_{3 \times 4}(K_{60 \times 9 + 1})$ we need. \square

LEMMA 2.10. *There exists a $D_{3 \times 4}(K_{601})$.*

Proof. By Lemma 2.2, we know that a $D_{3 \times 4}(K_5(15))$ exists. Since an $OA(8, 5)$ exists, we get a $D_{3 \times 4}(K_5(120))$ by Lemma 1.6. This implies that we have a $D_{3 \times 4}(K_{60 \times 10 + 1})$ (by Lemmas 1.4 and 2.3). \square

LEMMA 2.11. *There exists a $D_{3 \times 4}(K_{661})$.*

Proof. We know that a 5-GDD of type 4^{11} exists; see [7] for a reference. Therefore, a $D_{3 \times 4}(K_{11}(60))$ can be obtained by inflating each partite set to 60 points. Since a $D_{3 \times 4}(K_{61})$ exists, we get a $D_{3 \times 4}(K_{60 \times 11 + 1})$ by Lemma 1.4. \square

THEOREM 2.12. *There exists a $D_{3 \times 4}(K_{60t+1})$ for each positive integer t .*

Proof. When $t \geq 12$, we know that a $GD[K, 1, M; t]$ exists, where $K = \{4, 5\}$ and $M = \{1, 2, \dots, 7\}$ by [11, Lemma 3.2]. We also know that a $D_{3 \times 4}(K_{60m+1})$ for any $m \in M$ and a $D_{3 \times 4}(K_k(60))$ for any $k \in K$ exist by the above lemmas. So, by Lemma 1.5, a $D_{3 \times 4}(K_{60t+1})$ exists when $t \geq 12$. Combining the results of Lemma 2.3 to Lemma 2.11, we conclude the proof. \square

2.2. $v \equiv 21 \pmod{60}$. By Lemma 1.4, it suffices to prove that a $D_{3 \times 4}(K_{21})$ exists and a $D_{3 \times 4}(K_{3t+1}(20))$ exists for each positive integer t .

LEMMA 2.13. *There exists a $D_{3 \times 4}(K_{21})$.*

Proof. The design is constructed on $V = Z_7 \times \{0, 1, 2\}$, generated by the group Z_7 acting on the points. The base grid-block is the following 3×4 array:

(0,0)	(1,0)	(3,0)	(0,1)
(2,1)	(4,1)	(1,1)	(0,2)
(3,2)	(6,2)	(5,2)	(6,0)

□

LEMMA 2.14. *There exist both a $D_{3 \times 4}(K_7(20))$ and a $D_{3 \times 4}(K_{10}(20))$.*

Proof. A $D_{3 \times 4}(K_7(20))$ is constructed on the Abelian group $V = Z_{140}$ generated by the group V acting on the points. This design has two base grid-blocks:

0	1	3	9
4	14	19	31
23	61	90	123

0	11	45	86
37	101	81	7
88	70	113	31

A $D_{3 \times 4}(K_{10}(20))$ is constructed on the Abelian group $V = Z_{200}$ generated by the group V acting on the points. This design has three base grid-blocks:

199	74	46	122
38	195	103	141
132	181	90	159

44	69	78	133
139	110	54	52
55	38	192	126

61	53	2	8
96	192	18	91
73	74	70	106

□

LEMMA 2.15. *There exists a $D_{3 \times 4}(K_{19}(20))$.*

Proof. In order to obtain a $D_{3 \times 4}(K_{19}(20))$, we first construct a $D_{3 \times 4}(K_6(60))$, and then together with a $D_{3 \times 4}(K_4(20))$ we obtain the desired design.

First, we know that a 5-GDD of type 4^6 exists; see [7] for a reference. Then we inflate each point in each partite set to 15 points. So, we have a $D_{3 \times 4}(K_6(60))$ by obtaining 75 grid-blocks for each K_5 in a K_5 -design of $K_6(4)$.

Next, in order to construct the desired $D_{3 \times 4}(K_{19}(20))$, we first partition each partite set of $K_6(60)$ into three subsets (each of size 20). Then, we add 20 new points to $K_6(60)$. We view these 20 points as a subset A which is different from the subsets in $K_6(60)$. So, we have 19 subsets (each has 20 points) altogether. By Lemma 2.2, since a $D_{3 \times 4}(K_4(20))$ exists, we have a $D_{3 \times 4}(K_{19}(20))$ as desired. □

LEMMA 2.16. *For each positive integer t , a $D_{3 \times 4}(K_{3t+1}(20))$ exists.*

Proof. It is well known that a $(v, \{4, 7, 10, 19\}, 1)$ -PBD exists for each $v \equiv 1 \pmod{3}$ and $v \geq 4$. Then, by inflating each point to 20 points and using Lemmas 2.2, 2.14, and 2.15, respectively, we conclude the proof. □

THEOREM 2.17. *For each positive integer t , a $D_{3 \times 4}(K_{60t+21})$ exists.*

Proof. By Lemmas 1.4, 2.13, and 2.16, the proof is complete. □

2.3. $v \equiv 36 \pmod{60}$.

LEMMA 2.18. *There exists a $D_{3 \times 4}(K_{36})$.*

Proof. The design is constructed on $V = Z_{18} \times \{0, 1\}$, generated by the group Z_{18} acting on the points. Note that there are two base grid-blocks. The first one is of short orbit, which goes one sixth of the cycle, and the second one is of full orbit.

(0,0)	(9,0)	(0,1)	(9,1)
(6,0)	(15,0)	(6,1)	(15,1)
(12,0)	(3,0)	(12,1)	(3,1)

(0,0)	(1,0)	(4,0)	(2,1)
(2,0)	(12,0)	(17,1)	(1,1)
(12,1)	(5,1)	(9,0)	(15,1)

□

The following lemmas are essential for our direct constructions.

LEMMA 2.19. *There exists a $D_{3 \times 4}(K_7(10))$.*

Proof. The design is constructed on the Abelian group $V = Z_{70}$, generated by the group V acting on the points. It has one base grid-block:

1	2	4	10
5	32	55	43
23	7	38	67

□

In what follows, if a complete multipartite graph with t_i partite sets of size s_i for $i = 1, 2, \dots, h$, can be decomposed into $G(r, c)$'s, then we say that an $r \times c$ grid-block design of type $\prod_{i=1}^h s_i^{t_i}$ exists.

LEMMA 2.20. *There exists a 3×4 grid-block design of type $10^6 5^1$.*

Proof. Let $V = Z_{60} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. The design is constructed on the group V generated by the group Z_{60} acting on the points. The five infinite points remain fixed under the action of Z_{60} . It has one base grid-block:

0	1	3	11
16	36	31	2
33	14	10	∞

We remark here that the above grid-block has an extra property that all elements in the same row or same column with ∞ are in different residue classes (mod 5). Therefore, for the 12 grid-blocks generated by adding $i, 5+i, \dots, 55+i$, $i = 0, 1, 2, 3, 4$, to every point in each base grid-block, respectively, we use ∞_i for ∞ . Then we have the desired design. □

LEMMA 2.21. *There exists a $D_{3 \times 4}(K_6(10))$.*

Proof. Let the last partite set of $K_6(10)$ be $A = \{\infty_0, \infty_1, \infty_2, \dots, \infty_8, \infty_9\}$ and $V = Z_{50} \cup A$. The design is constructed on the group V generated by the group Z_{50} acting on the points. The 10 infinite points remain fixed under the action of Z_{50} . It has one base grid-block:

0	1	4	38
27	3	∞^*	21
19	12	40	∞

Now, similar to Lemma 2.20, we replace ∞^*, ∞ by ∞_i, ∞_{i+5} , respectively, and $i = 0, 1, 2, 3, 4$, for the corresponding grid-blocks. □

LEMMA 2.22. *There exists a $D_{3 \times 4}(K_{12}(10))$.*

Proof. Let the last partite set of $K_{12}(10)$ be $\{\infty_1, \infty_2, \dots, \infty_{10}\}$ and $V = Z_{110} \cup \{\infty_1, \infty_2, \dots, \infty_{10}\}$. Then, the design is constructed on the group V generated by Z_{110} acting on the points. Furthermore, the 10 infinite points remain fixed under the action of Z_{110} . It has two base grid-blocks:

0	1	3	7	0	10	45	73
5	17	∞_1	26	38	68	20	88
13	40	54	∞_6	70	44	101	27

□

LEMMA 2.23. *There exist a $D_{3 \times 4}(K_{13}(10))$ and a $D_{3 \times 4}(K_{13}(30))$, as well as a $D_{3 \times 4}(K_5(30))$.*

Proof. A $D_{3 \times 4}(K_{13}(10))$ is constructed on the Abelian group $V = Z_{130}$ generated by the group V acting on the points. This design has two base grid-blocks:

29	103	34	111	82	81	9	51
108	54	40	22	67	83	94	117
65	44	69	41	89	43	106	126

Thus, a $D_{3 \times 4}(K_{13}(30))$ can be obtained by applying Lemma 1.6 and by the existence of an $OA(3, 4)$.

A $D_{3 \times 4}(K_5(30))$ is constructed on the Abelian group $V = Z_{150}$ generated by the group V acting on the points. It has two base grid-blocks:

66	58	140	54	134	71	42	10	□
118	30	84	7	123	140	99	81	
87	96	103	60	6	57	55	9	

LEMMA 2.24. *There exists a 3×4 grid-block of type $30^4 15^1$.*

Proof. Let $V = Z_{120} \cup \{\infty_0, \infty_1, \infty_2, \dots, \infty_{13}, \infty_{14}\}$. The design is constructed on the group V generated by the group Z_{120} acting on the points. The 15 infinite points remain fixed under the action of Z_{120} . Then, it has two base grid-blocks:

0	1	3	10	0	17	62	43
5	16	∞	39	53	80	7	102
18	51	57	∞^s	90	59	21	∞^*

Now, by replacing three infinite points $(\infty, \infty^s, \infty^*)$ with $(\infty_i, \infty_{i+5}, \infty_{i+10})$, respectively, as above (Lemma 2.20), we have the desired design. □

LEMMA 2.25. *There exist a $D_{3 \times 4}(K_6(12))$, a $D_{3 \times 4}(K_{11}(12))$, and a $D_{3 \times 4}(K_{16}(4))$. Hence, there exist also a $D_{3 \times 4}(K_6(36))$, a $D_{3 \times 4}(K_{11}(36))$, and a $D_{3 \times 4}(K_{16}(36))$.*

Proof. A $D_{3 \times 4}(K_6(12))$ is constructed on the Abelian group $V = Z_{72}$ generated by the group V acting on the points. The base grid-block is:

4	38	25	65
0	7	53	50
14	5	30	13

A $D_{3 \times 4}(K_{11}(12))$ is constructed on the Abelian group $V = Z_{132}$ generated by the group V acting on the points. This design has two base grid-blocks:

120	48	52	11	70	73	108	58
57	110	78	109	9	48	113	1
44	2	38	125	16	46	65	75

A $D_{3 \times 4}(K_{16}(4))$ is constructed on the Abelian group $V = Z_{64}$ generated by the group V acting on the points. The base grid-block is as follows:

0	1	3	7
5	18	57	42
43	51	21	62

Since both an $OA(3, 4)$ and an $OA(9, 4)$ exist, the other desired designs can be obtained by applying Lemma 1.6. □

LEMMA 2.26. *There exist a $D_{3 \times 4}(K_9(5))$ and a $D_{3 \times 4}(K_9(20))$.*

Proof. The first design is constructed on $V = Z_{15} \times \{0, 1, 2\}$ generated by the group Z_{15} acting on the points. Here, the nine partite sets are $\{\{0+i, 3+i, 6+i, 9+i, 12+i\} : i = 0, 1, 2\} \times \{0, 1, 2\}$. The base grid-block is the following:

(0,0)	(1,0)	(5,0)	(0,1)
(8,0)	(4,1)	(2,1)	(0,2)
(1,1)	(8,1)	(11,2)	(10,0)

(0,0)	(2,0)	(14,2)	(4,2)
(4,1)	(10,1)	(9,1)	(6,2)
(3,2)	(2,2)	(10,1)	(8,0)

Now, by using an $OA(4, 4)$, we can obtain a $D_{3 \times 4}(K_9(20))$. □

LEMMA 2.27. *There exists a $D_{3 \times 4}(K_{12}(5))$.*

Proof. Let $V = Z_{55} \cup \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4\}$. The design is constructed on the group V generated by the group Z_{55} acting on the points. The five infinite points remain fixed under the action of Z_{55} . The only base grid-block is the following:

0	1	3	7
5	20	28	48
29	11	45	∞

The above grid-block has the extra property that the all elements in the same row or same column with ∞ are in different residue classes (mod 5). Then, similar to Lemma 2.20, we have the construction. □

In order to properly combine the above lemmas together, we also need the following crucial theorem.

THEOREM 2.28 (see [2, 5, 7]).

1. A 5-GDD of type 4^{n+1} exists for $n \equiv 0, 4 \pmod{5}$.
2. A 5-GDD of type $4^n 8^1$ exists for $n \equiv 0, 2 \pmod{5}$ and $n \geq 7$, except possibly when $n = 10$.
3. A 5-GDD of type $4^n 12^1$ exists for $n \equiv 0 \pmod{5}$ and $n \geq 10$.
4. A 5-GDD of type $4^n 16^1$ exists for $n \equiv 0, 3 \pmod{5}$ and $n \geq 13$, except possibly when $n = 15$.
5. A 5-GDD of type $4^n 20^1$ exists for $n \equiv 0, 1 \pmod{5}$ and $n \geq 16$.
6. A 5-GDD of type $4^n 24^1$ exists for $n \equiv 0, 4 \pmod{5}$ and $n \geq 19$.
7. There exist 5-GDDs of types $8^5 4^1$ and $8^3 4^7$.
8. A 7-GDD of type n^7 exists when $n \geq 7$ is a prime power and $n = 12$.
9. There exist 7-GDDs of types $6^8, 6^{15}, 6^{28}$, and 6^{29} .

LEMMA 2.29. *There exists a $D_{3 \times 4}(K_{60n+36})$ for each odd positive integer n , where $n \notin \{1, 5\}$.*

Proof. We start with the constructions of small 3×4 grid-block designs. First, if $n = 3$, then K_{60n+36} is the union of $K_6(36)$ and six K_{36} 's. By Lemmas 2.18 and 2.25, we have a $D_{3 \times 4}(K_{216})$. Similarly, for $n = 9$, since K_{576} is the union of $K_{16}(36)$ and 16 K_{36} 's, we have the construction by Lemmas 2.18 and 2.25. For $n = 15, 21$, a $D_{3 \times 4}(K_{60n+36})$ can also be obtained by a similar argument. Here, we need a $D_{3 \times 4}(K_5(180))$ and a $D_{3 \times 4}(K_7(180))$, which can be obtained by inflating a $D_{3 \times 4}(K_5(15))$ (Lemma 2.2) and a $D_{3 \times 4}(K_7(20))$ (Lemma 2.14) with an $OA(12, 4)$ and an $OA(9, 4)$, respectively.

Now we consider the case $n = 7$. By Theorem 2.28, a 7-GDD of type 6^8 exists. Therefore, we may apply a basic technique of combinatorial designs to this GDD by assigning weights to the points (If a point is of weight h , then we inflate the point into h points.). Let each point in the first seven groups of size 6 be assigned weight 10, and let the weights of the points in the last group be $(5, 5, 5, 5, 5, 10)$. Then, a 3×4 grid-block design of type $60^7 35^1$ exists. This is by the fact that a $D_{3 \times 4}(K_7(10))$ exists (Lemma 2.19) and that a 3×4 grid-block design of type $10^6 5^1$ exists (Lemma 2.20). By using Lemma 1.4 and the existence of a $D_{3 \times 4}(K_{61})$ and a $D_{3 \times 4}(K_{36})$, we have the construction of a $D_{3 \times 4}(K_{60 \times 7 + 36})$.

For $n = 11$, take a 7-GDD of type 11^7 (Theorem 2.28) and remove a point. Use the deleted point to redefine the groups and adjoin an extra point to obtain a $\{7, 12\}$ -GDD of type $6^{11}12^1$, where the blocks of size 12 meet exactly one single point x in the group of size 12. Give weight 10 to points in the group of size 6 and the point x , and give weights 0, 5, or 10 to the remaining points of the group of size 12. Since we have 3×4 grid-block designs of types 10^6 (Lemma 2.21), $10^6 5^1$ (Lemma 2.20), 10^7 (Lemma 2.19), and 10^{12} (Lemma 2.22), we get a 3×4 grid-block design of type $60^{11}35^1$. Add an extra point to obtain the desired design. Similarly, by using a 7-GDD of type 13^7 (Theorem 2.28), we are able to construct a $D_{3 \times 4}(K_{60 \times 13 + 36})$. Here, we need a 3×4 grid-block design of type 10^{13} (Lemma 2.23).

For convenience in the constructions of a $D_{3 \times 4}(K_{60n+36})$ for larger n 's, we split the proof into three cases.

Case 1. $n \equiv 3, 7 \pmod{10}$, $n \geq 17$.

By Theorem 2.28, there exists a 5-GDD of type $4^t 8^1$ for each $t \equiv 0, 2 \pmod{5}$ and $t \geq 7$, $t \neq 10$. Now, for the group of size 8, we assign weights of (15, 15, 15, 30, 30, 30, 30, 30) to these points; all the points in the other groups are assigned weight 30. Therefore, by adding 21 points, we conclude that a $D_{3 \times 4}(K_{60n+36})$ exists. This is by the fact that there exists a $D_{3 \times 4}(K_{60k+21})$ for each $k \geq 0$ which has a subdesign $D_{3 \times 4}(K_{21})$ and, also, there exist a $D_{3 \times 4}(K_5(30))$, a $D_{3 \times 4}(K_{216})$, and a 3×4 grid-block design of type $30^4 15^1$ (Lemma 2.24). Since $t \neq 10$, the case $n = 23$ has to be considered separately. By a similar technique with a 5-GDD of type 8^6 , this can be done.

Case 2. $n \equiv 1, 7, 9 \pmod{10}$, $n \geq 39$.

Apply a 5-GDD of type $4^k 20^1$ ($n = 2k + 7$ or $n = 2k + 9$); then we have the construction by using a similar technique as in Case 1.

Case 3. $n \equiv 5, 7, 9 \pmod{10}$, $n \geq 45$.

Apply a 5-GDD of type $4^k 24^1$ ($n = 2k + 7$ or $n = 2k + 9$) and a similar technique.

Now, we have to handle the small cases. Since the technique is similar, we simply list their corresponding 5-GDD, as constructed in Theorem 2.28:

n	Type
19	8^5
23	8^6
25	$8^3 4^7$
29	13^5

Types of 5-GDD used in construction.

We are left with $n = 31, 35$. For $n = 31$ and 35, we start from a 5-GDD of type 7^5 and give weight 0 to 3 or 4 points in one group, and weight 60 to other points. Then, we add 36 points and fill in the holes to obtain the desired designs, since we have a $D_{3 \times 4}(K_{60k+36})$ containing a $D_{3 \times 4}(K_{36})$ as a subdesign for $k = 3, 7$. Here, the input designs, a $D_{3 \times 4}(K_4(60))$, and a $D_{3 \times 4}(K_5(60))$ come from Lemmas 2.2 and 2.2, respectively. \square

LEMMA 2.30. *There exists a $D_{3 \times 4}(K_{60n+36})$ for each even positive integer, where $n \notin \{2, 4, 10, 20, 22, 26\}$.*

Proof. First, if $n = 6$, then a $D_{3 \times 4}(K_{396})$ can be obtained by a $D_{3 \times 4}(K_{36})$ (11 copies) and a $D_{3 \times 4}(K_{11}(36))$ (Lemma 2.25). For $n = 8$, we first use a 7-GDD of type 8^7 to obtain a 3×4 grid-block design of type $80^6 35^1$ by assigning appropriate weights (0 or 5 or 10). Then, we add one point to obtain the desired grid-block design. Similarly, if $n = 14, 28$, take a 7-GDD of type 6^{n+1} (Theorem 2.28) and assign weights

0 or 5 or 10 to points in one group and weight 10 to all the other points; then we obtain a 3×4 grid-block design of type $60^n 35^1$. Therefore, the construction follows by adding one more point properly. The case $n = 16$ can be obtained by taking a 7-GDD of type 16^7 , assigning weights as in the above case, and adding 21 points. Note that there exists a $D_{3 \times 4}(K_{181})$ which contains a subdesign $D_{3 \times 4}(K_{21})$; see Lemma 2.26. Finally, for $n = 34$, take a 5-GDD of type 7^5 and give weights 0 to 60 to points in one group and weight 60 to all the other points to obtain a 3×4 grid-block design of type $420^4 360^1$. Then, the design will be obtained by adding 36 points and filling in the holes with a $D_{3 \times 4}(K_{396})$ (see Lemma 2.29) and a $D_{3 \times 4}(K_{456})$ containing a $D_{3 \times 4}(K_{36})$ as a subdesign (see Lemma 2.29).

When $n = 12, 18, 24, 30, 36$, we first obtain a $D_{3 \times 4}(K_{10t+1}(36))$ for $t = 2, 3, 4, 5, 6$, and then fill in the holes with a $D_{3 \times 4}(K_{36})$ to obtain the desired design. For $t = 2, 6$, we start from a $D_{3 \times 4}(K_{10t+1}(1))$ and inflate by 36. For $t = 3$, take a $D_{3 \times 4}(K_6(12))$ from Lemma 2.25 and inflate by 15 to obtain a $D_{3 \times 4}(K_6(180))$. Then, add 36 points and fill in the holes with a $D_{3 \times 4}(K_6(36))$ (Lemma 2.25) to obtain a $D_{3 \times 4}(K_{31}(36))$. For $t = 4$, take a $D_{3 \times 4}(K_8(15))$ and inflate by 12. Then, fill in the holes with a $D_{3 \times 4}(K_6(36))$ to obtain a $D_{3 \times 4}(K_{41}(36))$. For $t = 5$, take a $D_{3 \times 4}(K_{10}(20))$ (Lemma 2.14), inflate by nine, and fill in the holes to obtain a $D_{3 \times 4}(K_{51}(36))$.

Now, consider larger n 's. If $n \equiv 2, 6 \pmod{10}$, $n \geq 32$, the construction of a $D_{3 \times 4}(K_{60n+36})$ can be obtained by taking a 5-GDD of type $4^k 16^1$ to obtain a 3×4 grid-block design of type $120^k 375^1$ (by assigning appropriate weights and adding 21 points). For $n \equiv 0, 4, 8 \pmod{10}$, we use a 5-GDD of type $4^k y^1$, where $y = 20$ or 24 for $n \geq 38$ by a similar construction. Therefore, we have the proof. \square

2.4. $v \equiv 16 \pmod{60}$. First, we show the nonexistence of order 16.

LEMMA 2.31. *There does not exist a $D_{3 \times 4}(K_{16})$.*

Proof. Suppose not. Since a $G(3, 4)$ is a 5-regular graph, each vertex of K_{16} is incident to exactly three grid-blocks. Moreover, since a $G(3, 4)$ has 30 edges, K_{16} can be decomposed into four grid-blocks. This implies that every grid-block B_i corresponds to a set S_i of four vertices which is vertex-disjoint from B_i , $i = 1, 2, 3, 4$, and also that $\{S_1, S_2, S_3, S_4\}$ forms a partition of $V(K_{16})$. Let $S_1 = \{a, b, c, d\}$. Then, the edges in $\langle S_1 \rangle$ (the subgraph of K_{16} induced by S_1) must be in B_2, B_3 , and (or) B_4 . Thus, $S_1 \subseteq V(B_i), i = 2, 3, 4$. Now, if the vertices of S_1 occur in a B_i (say B_2) which are in at most two rows, then let $\{e, f, g, h\}$ be the set of vertices which occur in a row of B_2 which contains no vertices of $\{a, b, c, d\}$. Clearly, the vertices in $\{e, f, g, h\}$ are in the grid-block B_1 . Since e, f, g , and h are in the same row of B_2 , $\langle \{e, f, g, h\} \rangle \cong K_4$ and, therefore, every edge of this graph cannot be an edge in B_1 , i.e., they must be in distinct rows and columns. But, this is impossible, since we have only three rows. Thus, the vertices of S_1 occur in three rows of $B_i, i = 2, 3, 4$. Hence, B_2, B_3 , and B_4 contain at least one edge of $\langle S_1 \rangle$, respectively. Without loss of generality, let B_2 contain the maximum number of edges in $\langle S_1 \rangle$ and let B_4 contain the minimum number of edges in $\langle S_1 \rangle$. Therefore, we have the following cases to consider: (i) $\langle 4, 1, 1 \rangle$ (see Figure 1), (ii) $\langle 3, 2, 1 \rangle$ (see Figure 2), and (iii) $\langle 2, 2, 2 \rangle$ (see Figure 3), where the first coordinate represents the number of edges in $\langle S_1 \rangle$ which are in B_2 and so on.

- (i) $\langle 4, 1, 1 \rangle$. Here, consider $*$ in B_3 ; let it be x . Since x must be incident with every vertex in $V(K_{16}) \setminus \{x\}$, x has to be in B_2 or B_4 such that x is incident to a and no other vertices of $\{b, c, d\}$. But, clearly this is not possible.
- (ii) $\langle 3, 2, 1 \rangle$. Similarly, consider $*$ in B_3 and the vertices in B_2 or B_4 which are incident to c and also incident to one of the vertices in $\{a, b, d\}$. This is not possible either.

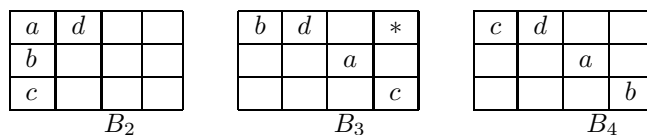


FIG. 1. Obtained by permuting rows, columns, and/or vertices in $\{a, b, c, d\}$.

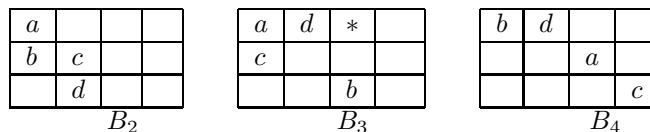


FIG. 2. Obtained by permuting rows, columns, and/or vertices in $\{a, b, c, d\}$.

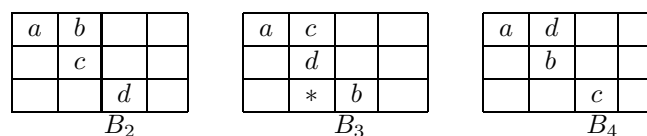


FIG. 3. Obtained by permuting rows, columns, and/or vertices in $\{a, b, c, d\}$.

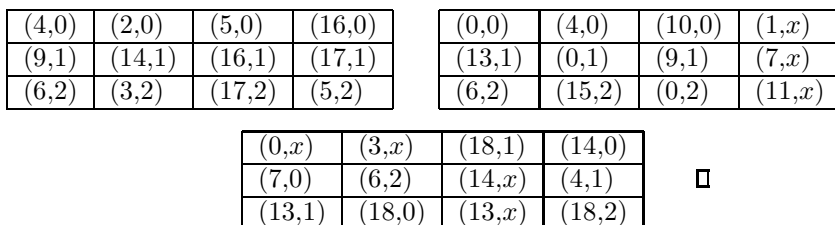
(iii) $\langle 2, 2, 2 \rangle$. Similarly, consider $*$ in B_3 and the vertices in B_2 or B_4 which are incident to a and also incident to a vertex in $\{b, c, d\}$. This case is not possible either. Since (after all cases are considered) there are no suitable decompositions obtained, we conclude that no $D_{3 \times 4}(K_{16})$ exists. \square

LEMMA 2.32. *There exists a $D_{3 \times 4}(K_{76})$.*

Proof. Let the point set be $V = Z_{19} \times (Z_3 \cup \{x\})$. We view x as an infinite point, that is, we consider $x + 1 = x$. It is easy to see that 7 is the cube root of unity in Z_{19} . Now, we can define the automorphism group to be a non-Abelian group generated by the following two automorphisms:

$$\sigma : (a, b) \longrightarrow (a + 1, b), \tau : (a, b) \longrightarrow (7a, b + 1).$$

Here, the order of σ is 19 and that of τ is 3. Hence, the automorphism group is of order 57 and the design has three base grid-blocks; the third one is of full orbit and the other two are of short orbits.



LEMMA 2.33. *There exists a $D_{3 \times 4}(K_{13}(15))$.*

Proof. The design is constructed on the Abelian group $V = Z_{195}$ generated by the group V acting on the points. Since 16 is the cube root of unity in Z_{195} , we can first find one base grid-block and get all three base grid-blocks by multiplying 1, 16, and 61, respectively. Here is the first one:

0	1	3	7
5	15	23	147
59	166	50	185

□

LEMMA 2.34. *There exists a $D_{3 \times 4}(K_{60n+76})$ for each even positive integer n , where $n \notin \{2, 6, 22\}$.*

Proof. Clearly, if there exists a 5-GDD with at least one group of size 4, then we give weight 15 to three points in the group and weight 30 to all the other points in the 5-GDD. Now, we have a 3×4 grid-block GDD with a group of size 75 and all the other groups of size a multiple of 120, provided that each group in the 5-GDD is of size a multiple of 4. Now, by adding one point to this inflated GDD and filling in the holes with proper 3×4 grid-block designs, we are able to obtain the desired designs. So, it is not difficult to see that by applying a 5-GDD of type 4^{k+1} , where $n = 2k$ and $n \geq 8$, we have the cases $n \equiv 0, 8 \pmod{10}$. Then, by applying a 5-GDD of type $4^k 8^1$, where $n = 2(k + 1)$ and $n \geq 16$, we have the cases $n \equiv 2, 6 \pmod{10}$ except when $n = 22$. Finally, we apply a 5-GDD of type $4^k 12^1$, where $n = 2(k + 2)$ and $n \geq 24$, and we have the cases $n \equiv 4 \pmod{10}$. Combining the results obtained above, we have the cases $n = 4, 12, 14$ remaining.

For $n = 4$, we first obtain a $D_{3 \times 4}(K_9(35))$ by assigning weight 7 to each point of a $D_{3 \times 4}(K_9(5))$ (Lemma 2.26). Then, the case $n = 4$ follows by adding a point and using a $D_{3 \times 4}(K_{36})$ to fill in the holes.

For $n = 12$, take a 7-GDD of type 12^7 , give weight 10 to six points in one specific group, weight 5 to one point in the same group, and weight 0 to the remaining five points of the group. Moreover, we give weight 10 to all the points in the other six groups. Now, add 10 points and fill in the six groups mentioned above with a $D_{3 \times 4}(K_{13}(10))$ from Lemma 2.23. Now, by using one point with weight 0 to redefine the groups, we have a 3×4 grid-block design of type $60^{12} 75^1$, and the case $n = 12$ follows by adding one more point and filling in the holes with a $D_{3 \times 4}(K_{61})$ and a $D_{3 \times 4}(K_{76})$, respectively.

Finally, for $n = 14$, take a 7-GDD of type 6^{15} and give weight 5 to one point and 10 to all the other points. Then, we have a 3×4 grid-block design of type $60^{14} 55^1$. Therefore, we have the construction by adding 21 points and filling in the holes with a $D_{3 \times 4}(K_{81})$ containing a $D_{3 \times 4}(K_{21})$ as a subdesign (see Theorem 2.17) and a $D_{3 \times 4}(K_{76})$, respectively. □

LEMMA 2.35. *There exists a $D_{3 \times 4}(K_{60n+76})$ for each odd positive integer n , where $n \notin \{1, 3, 9, 17, 21\}$.*

Proof. Since a 5-GDD of type 5^q , where $q \equiv 1 \pmod{4}$, exists [5] and a $D_{3 \times 4}(K_5(15))$ exists, we have a $D_{3 \times 4}(K_q(75))$ by giving weight 15. Now, by adding one point and filling in the holes with a $D_{3 \times 4}(K_{76})$, we obtain a $D_{3 \times 4}(K_{60n+76})$ for each $n \equiv 0 \pmod{5}$ and $n \geq 5$. Note that $60n + 76 = 1 + (300k + 75) = 1 + 75q$, where $q = 4k + 1$. By a similar idea to that used in Lemma 2.34, we take a 5-GDD of type $4^k 16^1$ (Theorem 2.28), and obtain a 3×4 grid-block design of type $120^k 375^1$ by giving weight 30 to points in the groups of size 4 and weight 15 or 30 to points in the group of size 16. This implies that we have the cases $n \equiv 1 \pmod{10}$, $n \geq 31$, by adding one more point. Also, take a 5-GDD of type $4^k 20^1$ to obtain a 3×4 grid-block design of type $120^k x^1$ with $x = 60 \times 5 + 75$ or $x = 60 \times 7 + 75$ and a 5-GDD of type $4^k 24^1$ to obtain a 3×4 grid-block design of type $120^k 375^1$ by assigning appropriate weights. Add one point and fill in the holes to settle the cases $n \equiv 7, 9 \pmod{10}$, $n \geq 37$ and $n \equiv 3 \pmod{10}$, $n \geq 43$. Here, we need a $D_{3 \times 4}(K_{60 \times 7 + 75})$ as the ingredient design. Take a 7-GDD of type 6^8 to obtain a 3×4 grid-block design of type $60^7 55^1$ and add

21 points to finish off the construction.

Now, it remains to consider the small cases. First, for $n = 27$, take a 7-GDD of type 6^{28} to obtain a 3×4 grid-block design of type $60^{27}55^1$ and add 21 points. For $n = 11$, we give weight 35 to each point of a $D_{3 \times 4}(K_{21})$ to obtain a $D_{3 \times 4}(K_{21}(35))$ and add one point. For convenience, we list the rest of the cases.

- $n = 13$. Take a 7-GDD of type 13^7 to obtain a 3×4 grid-block design of type $60^{13}75^1$ first, and then add one point.
- $n = 19$. By the existence of $D_{3 \times 4}(K_{16}(4))$ (Lemma 2.25), we have a $D_{3 \times 4}(K_{16}(76))$. Then, fill in the holes with a $D_{3 \times 4}(K_{76})$.
- $n = 23, 29$. First, we construct a $D_{3 \times 4}(K_{n+1}(10))$. For $n = 23$, we use the existence of a $D_{3 \times 4}(K_4(60))$ and for $n = 29$, we use a $D_{3 \times 4}(K_5(60))$ and then fill in the holes of size 60 with a $D_{3 \times 4}(K_6(10))$. Then, take a 7-GDD of type n^7 , give weight 10 to six points in one specific group, weight 5 to one point in the same group, and weight 0 to the remaining five points of the group. Moreover, we give weight 10 to all the points in the other six groups. Now, add 10 points and fill in the six groups mentioned above with a $D_{3 \times 4}(K_{n+1}(10))$. Now, by using one point with weight 0 to redefine the groups, we have a 3×4 grid-block design of type $60^n 75^1$, and the existence of $n = 23, 29$ follows by adding one more point and filling in the holes with a $D_{3 \times 4}(K_{61})$ and a $D_{3 \times 4}(K_{76})$, respectively.
- $n = 33$. Take a 5-GDD of type 7^5 (Theorem 2.28) and give weight 60 to all but two points (weight 0) in one group. Then, we obtain a 3×4 grid-block design of type $420^4 300^1$. The proof follows by adding 76 points. \square

To conclude this section, we would like to point out that the existence of a $D_{3 \times 4}(K_{60 \times 21 + 76})$ can be obtained by using transversal designs with holes.

Let S be a set of size s , and let $\mathcal{H} = \{S_1, S_2, \dots, S_n\}$ be a set of subsets of S . A holey Latin square having hole set \mathcal{H} is an $s \times s$ array L , whose rows and columns are indexed by elements of S and possess the following further properties:

1. Each cell in L is either empty or contains an element of S .
2. Every element of S appears at most once in any row or column of L .
3. The subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq n$ (these subarrays are referred to as holes).
4. Symbol $s \in S$ occurs in row or column t if and only if $(s, t) \in (S \times S) \setminus \cup_{1 \leq i \leq n} (S_i \times S_i)$.

L is said to have order s ; if S_1, S_2, \dots, S_n are disjoint, it is also said to have type $(|S_1|, |S_2|, \dots, |S_n|)$. Alternatively, if for $1 \leq i \leq m$ there are u_i holes of size t_i , then we can write the type of S as $t_1^{u_1}, t_2^{u_2}, \dots, t_m^{u_m}$.

Two holey Latin squares of the same type on the same set S of size s and hole set \mathcal{H} are said to be orthogonal if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{1 \leq i \leq n} (S_i \times S_i)$. A set of k holey Latin squares is said to be orthogonal if every pair of them is orthogonal. It is not difficult to notice that the existence of k holey mutually orthogonal Latin squares of type g^n is equivalent to the existence of a $(k + 2)$ -HTD (holey transversal design) of type g^n .

LEMMA 2.36. *If there exists a resolvable 6-HTD of type g^n , and there exists a 3×4 grid-block design of type $(10g)^n x^1$, then there exists a 3×4 grid-block design of type $(60g)^n (x + y)^1$, where $0 \leq y \leq 10g(n - 1)$ and $x + y$ is admissible.*

Proof. Since a resolvable 6-HTD has $g(n - 1)$ parallel classes of blocks, we can add $g(n - 1)$ extra points to extend the blocks of these parallel classes. Assign weights 0, 5, or 10 to these new $g(n - 1)$ points and let the weight of each point in the 6-HTD be 10. Here, the input designs 3×4 grid-block designs of types 10^6 , $10^6 5^1$, and 10^7 from

Lemmas 2.21, 2.20, 2.19, respectively. Then, by adjoining x additional points and filling in the groups from the HTD with a 3×4 grid-block design of type $(10g)^n x^1$, we obtain the desired 3×4 grid-block design of type $(60g)^n(x + y)^1$. \square

LEMMA 2.37. *There exists a $K_{3 \times 4}(K_{60n+76})$ when $n = 21$.*

Proof. Apply Lemma 2.36 with a resolvable 6-HTD of type 3^7 together with a 3×4 grid-block design of type 30^7 to obtain a 3×4 grid-block design of type $180^7(0 + 75)^1$. Then, adjoin an extra point and fill in the holes to obtain the desired design. Here, the resolvable 6-HTD of type 3^7 comes from [1, Theorem 3.4]. The existence of a $D_{3 \times 4}(K_7(10))$ (Lemma 2.19) implies that of a $D_{3 \times 4}(K_7(30))$. \square

We summarize these results in this section with the following theorem.

THEOREM 2.38. *A $D_{3 \times 4}(K_v)$ exists if and only if $v \equiv 1, 16, 21, 36 \pmod{60}$ except for when $v = 16$ and possibly except when $v \in \{60n+36 | n = 1, 2, 4, 5, 10, 20, 22, 26\} \cup \{60n + 16 | n = 2, 3, 4, 7, 10, 18, 23\}$.*

3. 4×4 grid-block designs. By Lemma 1.1, it is easy to find a necessary condition for the existence of a $D_{4 \times 4}(K_v)$.

LEMMA 3.1. *If a $D_{4 \times 4}(K_v)$ exists, then $v \equiv 1 \pmod{96}$.*

In what follows, we prove that the above necessary condition is also sufficient by constructing a $D_{4 \times 4}(K_v)$ for each $v \equiv 1 \pmod{96}$. We start with the smallest nontrivial case: $v = 97$.

LEMMA 3.2. *There exists a $D_{4 \times 4}(K_{97})$.*

Proof. By a computer search, we obtain a base grid-block. (There are exactly 48 distinct differences, $1, 2, \dots, 48$, which are obtained from the four rows and four columns of the array.)

0	1	3	7
5	13	81	38
16	60	26	86
46	74	61	29

□

LEMMA 3.3. *There exists a $D_{4 \times 4}(K_4(4))$.*

Proof. Let the four partite sets of $K_4(4)$ be $A_0 = \{0, 4, 8, 12\}$, $A_1 = \{1, 5, 9, 13\}$, $A_2 = \{2, 6, 10, 14\}$, and $A_3 = \{3, 7, 11, 15\}$. Then a $D_{4 \times 4}(K_4(4))$ contains exactly two 4×4 grid-blocks as follows:

0	1	2	3	0	9	14	17
5	4	7	6	13	4	3	10
10	11	8	9	6	15	8	1
15	14	13	12	11	2	5	12

□

We note here that if $K_m(n)$ has a decomposition into subgraphs K_4 , then we have a GDD, $GD[4, 1, n; mn]$; see [3] for a reference. The following result has been known for some time.

THEOREM 3.4. *A $GD[4, 1, n; mn]$ exists if and only if*

1. $3|(m - 1)n$ and
2. $12|n^2m(m - 1)$.

By using Lemma 3.3 and Theorem 3.4, we can prove the following lemma.

LEMMA 3.5. *For each $m \geq 4$, a $D_{4 \times 4}(K_m(96))$ exists.*

Proof. First, by partitioning each partite set of $K_m(96)$ into 24 subsets (each of size 4), we have a new graph $G \cong K_m(24)$ in which each point represents a subset of size 4. Denote the m partite sets of G by A_0, A_1, \dots, A_{m-1} , where $A_i = \{A_{i,0}, A_{i,1}, \dots, A_{i,23}\}$, $i \in \mathbf{Z}_m$. Therefore, $V(G) = \{A_{i,j} | i \in \mathbf{Z}_m \text{ and } j \in \mathbf{Z}_{24}\}$.

By Theorem 3.4, a $GD[4, 1, 24; 24m]$ exists for each $m \geq 4$. Therefore, we conclude the proof by using Lemma 3.3 to obtain two grid-blocks for each K_4 in a K_4 -design of G . \square

LEMMA 3.6. *There exists a $D_{4 \times 4}(K_{193})$.*

Proof. First, we construct a base grid-block of a $D_{4 \times 4}(K_{193})$ using a computer; see the following array:

$$A: \begin{array}{|c|c|c|c|} \hline 0 & 1 & 3 & 7 \\ \hline 5 & 14 & 25 & 39 \\ \hline 35 & 72 & 131 & 62 \\ \hline 82 & 150 & 110 & 183 \\ \hline \end{array}$$

Now we check whether the array A satisfies the condition of Lemma 1.7. First, we have $q = 193$, $e = 48$, and $f = (q - 1)/e = 4$. We let $F^* = GF(q) \setminus \{0\}$, ω be the primitive element of F^* (here $\omega = 5$), and $\varepsilon = \omega^e$. Second, we compute the cosets of C_0^e :

$C_0^e = \{1\ 112\ 192\ 81\}$	$C_1^e = \{5\ 174\ 188\ 19\}$
$C_2^e = \{25\ 98\ 168\ 95\}$	$C_3^e = \{125\ 104\ 68\ 89\}$
$C_4^e = \{46\ 134\ 147\ 59\}$	$C_5^e = \{37\ 91\ 156\ 102\}$
$C_6^e = \{185\ 69\ 8\ 124\}$	$C_7^e = \{153\ 152\ 40\ 41\}$
$C_8^e = \{186\ 181\ 7\ 12\}$	$C_9^e = \{158\ 133\ 35\ 60\}$
$C_{10}^e = \{18\ 86\ 175\ 107\}$	$C_{11}^e = \{90\ 44\ 103\ 149\}$
$C_{12}^e = \{64\ 27\ 129\ 166\}$	$C_{13}^e = \{127\ 135\ 66\ 58\}$
$C_{14}^e = \{56\ 96\ 137\ 97\}$	$C_{15}^e = \{87\ 94\ 106\ 99\}$
$C_{16}^e = \{49\ 84\ 144\ 109\}$	$C_{17}^e = \{52\ 34\ 141\ 159\}$
$C_{18}^e = \{67\ 170\ 126\ 23\}$	$C_{19}^e = \{142\ 78\ 51\ 115\}$
$C_{20}^e = \{131\ 4\ 62\ 189\}$	$C_{21}^e = \{76\ 20\ 117\ 173\}$
$C_{22}^e = \{187\ 100\ 6\ 93\}$	$C_{23}^e = \{163\ 114\ 30\ 79\}$
$C_{24}^e = \{43\ 184\ 150\ 9\}$	$C_{25}^e = \{22\ 148\ 171\ 45\}$
$C_{26}^e = \{110\ 161\ 83\ 32\}$	$C_{27}^e = \{164\ 33\ 29\ 160\}$
$C_{28}^e = \{48\ 165\ 145\ 28\}$	$C_{29}^e = \{47\ 53\ 146\ 140\}$
$C_{30}^e = \{42\ 72\ 151\ 121\}$	$C_{31}^e = \{17\ 167\ 176\ 26\}$
$C_{32}^e = \{85\ 63\ 108\ 130\}$	$C_{33}^e = \{39\ 122\ 154\ 71\}$
$C_{34}^e = \{2\ 31\ 191\ 162\}$	$C_{35}^e = \{10\ 155\ 183\ 38\}$
$C_{36}^e = \{50\ 3\ 143\ 190\}$	$C_{37}^e = \{57\ 15\ 136\ 178\}$
$C_{38}^e = \{92\ 75\ 101\ 118\}$	$C_{39}^e = \{74\ 182\ 119\ 11\}$
$C_{40}^e = \{177\ 138\ 16\ 55\}$	$C_{41}^e = \{113\ 111\ 80\ 82\}$
$C_{42}^e = \{179\ 169\ 14\ 24\}$	$C_{43}^e = \{123\ 73\ 70\ 120\}$
$C_{44}^e = \{36\ 172\ 157\ 21\}$	$C_{45}^e = \{180\ 88\ 13\ 105\}$
$C_{46}^e = \{128\ 54\ 65\ 139\}$	$C_{47}^e = \{61\ 77\ 132\ 116\}$

Third, by using the differences less than 97, we have $\vec{\partial}A = \{5, 30, 35, 47, 77, 82, 13, 58, 71, 78, 57, 44, 22, 87, 65, 21, 85, 86, 32, 23, 55, 72, 49, 17, 1, 2, 3, 4, 6, 7, 9, 11, 20, 14, 25, 34, 37, 59, 96, 69, 10, 27, 68, 40, 28, 73, 33, 92\}$. Therefore, by Lemma 1.7, a $D_{4 \times 4}(K_{193})$ exists. Note that all grid-blocks are obtained by calculating $cA+x$ for $c \in C_0^e/\{1, -1\} =$

$\{1, 112\}$ and $x \in GF(q)$. \square

LEMMA 3.7. *There exists a $D_{4 \times 4}(K_{289})$.*

Proof. By a similar idea, we find A first. Let $F^* = GF(17^2) \setminus \{0\}$ and ω be a primitive element of F^* ; here we let ω be the root of the primitive polynomial $\omega^2 + \omega + 3$. Then, we have the following:

A:

ω^∞	ω^0	ω^1	ω^2
ω^3	ω^4	ω^5	ω^6
ω^{20}	ω^{17}	ω^{155}	ω^{83}
ω^{46}	ω^{70}	ω^{221}	ω^7

Now, the cosets of C_0^e are as follows:

$C_0^e = \{1, \omega^{48}, \omega^{96}, \omega^{144}, \omega^{192}, \omega^{240}\}$
$C_1^e = \{\omega, \omega^{49}, \omega^{97}, \omega^{145}, \omega^{193}, \omega^{241}\}$
\vdots
$C_{46}^e = \{\omega^{46}, \omega^{94}, \omega^{142}, \omega^{190}, \omega^{238}, \omega^{286}\}$
$C_{47}^e = \{\omega^{47}, \omega^{95}, \omega^{143}, \omega^{191}, \omega^{239}, \omega^{287}\}$

Therefore, we can obtain the set of differences $\vec{\partial}A = \{\omega^3, \omega^8, \omega^{20}, \omega^{30}, \omega^{21}, \omega^{46}, \omega^{11}, \omega^{36}, \omega^5, \omega^7, \omega^{15}, \omega^{35}, \omega^{12}, \omega^{39}, \omega^{27}, \omega^{22}, \omega^{23}, \omega^{24}, \omega^{13}, \omega^{16}, \omega^{14}, \omega^{31}, \omega^{43}, \omega^{19}, 1, \omega^{37}, \omega, \omega^{38}, \omega^6, \omega^2, \omega^{40}, \omega^{41}, \omega^9, \omega^{42}, \omega^{10}, \omega^4, \omega^{18}, \omega^{45}, \omega^{33}, \omega^{29}, \omega^{28}, \omega^{47}, \omega^{26}, \omega^{25}, \omega^{32}, \omega^{44}, \omega^{34}, \omega^{17}\}$. Hence, we have a $D_{4 \times 4}(K_{289})$. Here, all grid-blocks are obtained by Lemma 1.7, calculating $cA + x$ for $c \in C_0^e / \{1, -1\}$ and $x \in GF(q)$. \square

THEOREM 3.8. *A $D_{4 \times 4}(K_v)$ exists if and only if $v \equiv 1 \pmod{96}$.*

Proof. By Lemmas 3.2 and 3.5, we know that a $D_{4 \times 4}(K_{96t+1})$ exists for each positive integer t , $t = 1$, and $t \geq 4$. So, by using Lemmas 3.6 and 3.7, we conclude that a $D_{4 \times 4}(K_v)$ exists for each $v \equiv 1 \pmod{96}$. Hence, by Lemma 3.1 the theorem is proved. \square

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