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Excessive near 1-factorizations

David Cariolaro [a,*], Hung-Lin Fu [b]

[a] *Institute of Mathematics, Academia Sinica, Nankang, Taipei 115, Taiwan*
[b] *Department of Applied Mathematics, National Chiao Tung University, Hsin Chu 30050, Taiwan*

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ABSTRACT

We begin the study of sets of near 1-factors of graphs G of odd order whose union contains all the edges of G and determine, for a few classes of graphs, the minimum number of near 1-factors in such sets.

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1. Introduction

We consider only finite, simple, undirected graphs. The vertex set, edge set and maximum degree of a graph G are denoted by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If every vertex has degree $d = \Delta(G)$, d is called the *degree* of G and G is said to be *d-regular*.

A *matching* M in a graph G is a set of mutually non-adjacent edges of G. A vertex $v \in V(G)$ is said to be *saturated* by M if it is the endpoint of an edge of M, and is said to be *unsaturated* or to *miss* M otherwise. A *1-factor* is a matching of G which saturates all vertices of G, and a *near 1-factor* is a matching M of G that saturates all vertices except one. A *1-factorization* (*near 1-factorization*) is a partition of $E(G)$ into disjoint 1-factors (near 1-factors).

If G and H are two graphs, the notation $G + H$ will be used to denote the *join* of G and H, i.e. the graph obtained by taking one isomorphic copy of G, one isomorphic copy of H, and joining every vertex of the copy of G to every vertex of the copy of H. If G is a graph and S is a set of vertices or edges of G, the notation $G - S$ will denote the graph obtained from G by deleting each element of S from G, together with all the edges incident with any vertex in S. The notations $G - v$ and $G - e$ will be shorthands for $G - \{v\}$ and $G - \{e\}$, respectively. For undefined notions we refer the reader to [3].

Bonisoli and Cariolaro [1] define *excessive factorization* of a graph G of even order to be a minimum set of (not necessarily disjoint) 1-factors of G whose union is $E(G)$. They denote by $\chi'_e(G)$ the number of 1-factors in an excessive factorization and call it *excessive index* of G. If G does not have an excessive factorization, the excessive index of G is conventionally set to ∞. In the same paper, Bonisoli and Cariolaro study excessive factorization of regular graphs as a natural extension of the concept of 1-factorization.

A few open questions and conjectures appear in [1]. In particular, the following conjecture seems particularly interesting and difficult. Recall that an *r-graph* is a regular graph G such that, if V_1 is a subset of $V(G)$ of odd cardinality, there are at

* Corresponding author.
E-mail addresses: cariolaro@math.sinica.edu.tw, davidcariolaro@hotmail.com (D. Cariolaro), hlfu@math.nctu.edu.tw (H.-L. Fu).

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least d edges in G joining V_1 to $V(G) \setminus V_1$, where d is the degree of G. (Notice that this implies that G has even order). The class of r-graphs is a generalisation of the class of 1-factorizable graphs, since it is easily seen that any 1-factorizable graph must be an r-graph.

Conjecture 1. *For every r-graph G of degree d,* $\chi'_e(G) \le d+2$.

This, if true, would be best possible because the Petersen graph is an r-graph, has degree 3 and (as proven in [1]) $\chi'_e(P) = 5$.

Graphs which admit an excessive factorization are easily characterized, since they coincide with those graphs for which every edge is in a 1-factor. We call such graphs *matching-covered*.

Notice that $\chi'_e(G)$ is defined to be ∞ if and only if G does not have an excessive factorization. In particular, $\chi'_e(G) = \infty$ if G has odd order. This disparity between graphs of even order and graphs of odd order is rather unpleasant.

In this respect, it may be interesting to replace (for graphs of odd order) 1-factors by *near 1-factors*. This is indeed the approach that we will take in this paper, and the following is the natural extension of the definition of excessive factorization for graphs of odd order.

Definition 1. An excessive near 1-factorization of a graph G is a minimum set of near 1-factors whose union contains all the edges of G.

If $\mathcal{F}$ is a set of near 1-factors of G whose union is $E(G)$, we call $\mathcal{F}$ a *near 1-factor cover*. The *size* of $\mathcal{F}$ is the number of near 1-factors in $\mathcal{F}$. Notice that an excessive near 1-factorization is just a near 1-factor cover of minimum size. When all the near 1-factors are mutually disjoint, we have an ordinary *near 1-factorization*. Thus, we may say that the concept of excessive near 1-factorization generalizes the concept of near 1-factorization.

Clearly, a graph G admits an excessive near 1-factorization if and only if G has odd order and every edge belongs to a near 1-factor of G. We shall still say, in this case (with a slight abuse of terminology), that G is *matching-covered*. Moreover, instead of adopting a new notation for the size of an excessive near 1-factorization, we shall keep the notation $\chi'_e(G)$ used for graphs of even order. Therefore, the definition of the parameter $\chi'_e(G)$ is now altered as follows:

$$\chi'_e(G) = \begin{cases} \min\{|\mathcal{F}| : \mathcal{F} \text{ is a 1-factor cover}\} & \text{if } G \text{ has even order, and} \\ \min\{|\mathcal{F}| : \mathcal{F} \text{ is a near 1-factor cover}\} & \text{if } G \text{ has odd order,} \end{cases}$$

with the convention that $\min \emptyset = \infty$.

Notice that there are no conflicts arising with previously used notations, since all graphs considered thus far (in this respect) have been graphs of *even* order.

2. Some preliminary facts

We denote the *chromatic index* of a graph G (i.e. the minimum number of colours in a proper edge colouring) by $\chi'(G)$. The following is an easy but useful inequality.

Proposition 1. *For any graph G,*

$$\chi'_e(G) \ge \chi'(G) \ge \max\left\{\Delta(G), \left\lceil \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor} \right\rceil\right\}.$$

Proof. The fact that $\chi'(G) \ge \Delta(G)$ is trivial. The fact that $\chi'(G) \ge \left\lceil \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor} \right\rceil$ follows from the fact that every colour class in an edge colouring contains at most $\lfloor |V(G)|/2 \rfloor$ edges. We now prove the first inequality. Let $\mathcal{F}$ be an excessive factorization or excessive near 1-factorization. Think of $\mathcal{F}$ as an edge multicolouring. Now delete from each edge (if necessary) some of the colours until each edge has a single colour. The result is clearly an edge colouring and we have used at most $\chi'_e(G)$ colours. Thus the required inequality follows. □

The following is a basic property of near 1-factorizations.

Proposition 2. *Let G be a graph of odd order and maximum degree Δ and assume G has a near 1-factorization $\mathcal{F}$. Then* $|\mathcal{F}| = \Delta$ *or* $\Delta + 1$. *Moreover* $|\mathcal{F}| = \Delta + 1$ *if and only if G is a complete graph.*

Proof. The fact that $|\mathcal{F}| \ge \Delta$ is obvious and follows directly from the definition of near 1-factorization. Suppose now that $|\mathcal{F}| \ge \Delta + 1$. Then

$$\begin{aligned} |E(G)| &= |\mathcal{F}| \frac{(|V(G)|-1)}{2} \ge (\Delta+1)\frac{(|V(G)|-1)}{2} \\ &= \frac{1}{2}(\Delta|V(G)| + |V(G)| - \Delta - 1) \ge \frac{1}{2}\Delta|V(G)| \ge |E(G)|, \end{aligned}$$

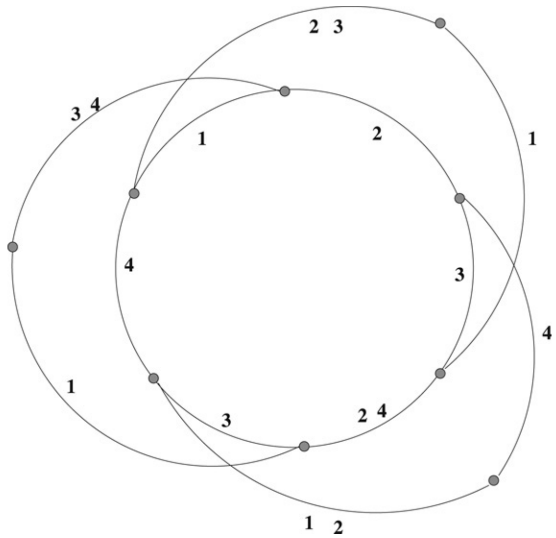

Fig. 1. An excessive near 1-factorization of P^*.

so that all the previous inequalities are equalities and, in particular, $|\mathcal{F}| = \Delta + 1$, $\Delta = |V(G)| - 1$ and G is Δ-regular, which implies that G is complete. Conversely, it is easy to see that the only near 1-factorization of the complete graph of odd order has size $\Delta + 1$, thus completing the proof. $\square$

A graph G is called *overfull* if $\Delta(G) < \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor}$. It is easy to see that every overfull graph has odd order. For a graph G and a vertex v, we let the *deficiency* of v be the integer $def(v) = \Delta(G) - \deg_G(v)$ and we let the *deficiency* of G, denoted $def(G)$, be defined as $def(G) = \sum_{v \in V(G)} def(v) = \Delta(G)|V(G)| - 2|E(G)|$. It is easy to see that a graph G of odd order satisfies $def(G) < \Delta(G)$ if and only if G is overfull. The class of graphs of odd order for which $def(G) = \Delta(G)$ (i.e. $\Delta(G) = \frac{2|E(G)|}{|V(G)|-1}$ is particularly interesting and its members will be named *Δ-deficient*.

Proposition 3. *Let G be a non-complete graph of odd order and maximum degree Δ and assume G has a near 1-factorization. Then G is Δ-deficient.*

Proof. Let $|V(G)| = 2n + 1$. We have (using Proposition 2)

$$|E(G)| = |\mathcal{F}|n = \Delta n$$

which implies that

$$def(G) = \Delta(2n + 1) - 2|E(G)| = \Delta. \quad \square$$

The converse of Proposition 3 does not hold, since e.g. the graph P^*, obtained by the Petersen graph by deleting one vertex, has odd order, maximum degree 3 and, as we shall prove below, does not have a near 1-factorization, but $def(P^*) = 3$.

It is easy to see that P^* has a near 1-factor cover consisting of 4 near 1-factors (see Fig. 1). Thus $\chi'_e(P^*) = 4$.

The following proposition characterizes near factorizations.

Proposition 4. *Let G be a graph of odd order and Δ-deficient. Add a new vertex v^+ and, for any vertex $w \in V(G)$, add exactly $def(w)$ edges between v^+ and w, obtaining a multigraph G^+. Then G^+ is 1-factorizable if and only if G is near 1-factorizable.*

Proof. Every near 1-factor of G can be extended to a 1-factor of G^+ using one of the edges v^+w. Conversely, every 1-factor of G^+ contains a near 1-factor of G. It is clear that every 1-factorization of G^+ induces a near 1-factorization of G and vice versa. $\square$

Using Proposition 4, we can easily see that P^* does not have a near 1-factorization, since, if P is the Petersen graph, then $P \cong (P^*)^+$, so that, if P^* had a near 1-factorization, by Proposition 4, P would have a 1-factorization, which is impossible.

A proof identical to the proof of Proposition 4 also shows that the complete graph K_{2n-1} has a near 1-factorization, which is induced by a 1-factorization of the complete graph K_{2n}. Therefore, $\chi'_e(K_{2n-1}) = \chi'_e(K_{2n}) = 2n - 1$.

A statement slightly more general than Proposition 4 is the following.

Proposition 5. *Let G be a graph of odd order. Then G is near 1-factorizable if and only if there exists a 1-factorizable multigraph G^+ such that $G = G^+ - v$.*

Proof. If G is near 1-factorizable, by Proposition 3, G is either complete or Δ-deficient. If it is complete then $G^+ = G + K_1$ is the required multigraph. If it is Δ-deficient, the graph G^+ of Proposition 4 is the required multigraph. For the converse, if G^+ is a 1-factorizable multigraph such that $G = G^+ - v$, then any 1-factorization of G^+ induces a near 1-factorization of G, so that G is near 1-factorizable. □

3. Excessive near 1-factorizations

If $\mathcal{F}$ is an excessive near 1-factorization, the set of vertices *missed by* $\mathcal{F}$ is defined to be the set of vertices missed by at least one element of $\mathcal{F}$. We start by proving a simple inequality.

Proposition 6. *Let G be a graph of even order and assume G is matching-covered. Let $v \in V(G)$. Then the graph $G - v$ is matching-covered and*

$$\chi'_e(G - v) \leq \chi'_e(G).$$

Proof. Let $\mathcal{F}$ be an excessive factorization of G. Let $\mathcal{F}'$ be the set of near 1-factors of $G - v$ obtained from $\mathcal{F}$ by deleting from each 1-factor in $\mathcal{F}$ the edge incident with v. Then it is easily seen that $\mathcal{F}'$ is a near 1-factor cover of $G - v$, which proves the desired assertion. □

Notice that the inequality of Proposition 6 can be strict. For example, if $G \cong K_1 + C_5$, and v is chosen so that $G - v \cong C_5$, then we have $\chi'_e(G) = 5$ and $\chi'_e(G - v) = 3$. However, in this example there is a trivial reason for $\chi'_e(G)$ to be larger than $\chi'_e(G - v)$, namely the fact that the vertex v is the *unique* vertex of large degree of G (which forces $\chi'_e(G)$ to be large by Proposition 1).

An example in which this phenomenon does not occur is the Petersen graph P. It is easy to see (and was formally proven in [1]) that $\chi'_e(P) = 5$, but, as we have shown earlier, the graph $P^* = P - v$ satisfies $\chi'_e(P^*) = 4$.

Thus, it is natural to ask the following question.

Question: For which graphs G of even order and vertices $v \in V(G)$, we have $\chi'_e(G) = \chi'_e(G - v)$?

This seems to be a difficult question in general. A partial answer will be given later. Here we notice the following fact, which follows easily from the definition, but is sometimes of some use.

Proposition 7. *Let G be a graph of even order and let $v \in V(G)$. Let $N(v)$ be the neighbourhood of v in G. Then we have $\chi'_e(G - v) = \chi'_e(G)$ if and only if $G - v$ has an excessive near 1-factorization whose set of missing vertices coincides with $N(v)$.*

Proof. Let $\mathcal{F}$ be an excessive near 1-factorization of $G - v$ and suppose that $\mathcal{F}$ misses precisely the vertices in $N(v)$. Then it is obvious that this excessive near 1-factorization extends to a 1-factor cover of G of the same size. But then, by Proposition 6, this 1-factor cover is necessarily an excessive factorization, which proves that $\chi'_e(G - v) = \chi'_e(G)$. Conversely, suppose that $\chi'_e(G - v) = \chi'_e(G)$. Let $\mathcal{F}$ be an excessive factorization of G. Then $\mathcal{F}$ induces a near 1-factor cover of $G - v$, with respect to which the set of missing vertices coincides with $N(v)$. By the assumption that $\chi'_e(G - v) = \chi'_e(G)$, this near 1-factor cover is necessarily an excessive near 1-factorization of $G - v$, concluding the proof. □

An easy consequence of the above proposition is the following.

Corollary 1. *Let H be a matching-covered graph of odd order. Let $\mathcal{F}$ be an excessive near 1-factorization of H. Let X be the set of vertices missed by $\mathcal{F}$. Let G be the graph obtained by adding a new vertex x to H and joining x to each vertex in X. Then G is matching-covered and $\chi'_e(G) = \chi'_e(H)$.*

In particular, the above corollary implies the following.

Corollary 2. *Let H be a matching-covered graph of odd order. There exists a matching-covered graph G of even order such that $H = G - v$ and $\chi'_e(G) = \chi'_e(H)$.*

Thus, in principle, the problem of evaluating $\chi'_e(H)$ is reduced to the problem of evaluating $\chi'_e(G)$ for a graph G of even order. Unfortunately, we are not able in general to use this fact since we do not know how to construct such a graph G (unless we construct first an excessive near 1-factorization of H).

The following proposition is also easy to prove but useful.

Proposition 8. *Let G be a graph of odd order and suppose that every matching extends to a near 1-factor. Then $\chi'_e(G) = \chi'(G)$.*

Proof. By Proposition 1 we have $\chi'_e(G) \geq \chi'(G)$. For the reverse inequality, consider any edge colouring of G with exactly $\chi'(G)$ colours. Then extend each colour class to a near 1-factor, thus obtaining a near 1-factor cover of size $\chi'(G)$, which proves $\chi'_e(G) \leq \chi'(G)$. □

For example, using Proposition 8 we have an independent proof of the fact that the complete graph K_{2n-1} satisfies $\chi'_e(K_{2n-1}) = 2n - 1$.

It is easy to see that, if G is regular and has even order, then $\chi'_e(G) = \Delta(G)$ if and only if G is 1-factorizable (indeed this fact was used in [1] to prove that the computation of $\chi'_e(G)$ for regular graphs is NP-hard). Proposition 3 suggests that, for graphs of odd order, the class of graphs corresponding to the regular graphs of even order, is the class of Δ-deficient graphs. Indeed we have the following.

Proposition 9. *Let G be a Δ-deficient graph of odd order. Then $\chi'_e(G) = \Delta(G)$ if and only if G has a near 1-factorization.*

Proof. Assume G has a near 1-factorization $\mathcal{F}$. Since G is not complete, by Proposition 2, $|\mathcal{F}| = \Delta(G)$. Since any near 1-factorization is an excessive near 1-factorization, we conclude that $\chi'_e(G) = \Delta(G)$. For the converse, assume that $\chi'_e(G) = \Delta(G)$. Let $\mathcal{F}$ be an excessive near 1-factorization of G. Since $|E(G)| = \Delta(G)\frac{|V(G)|-1}{2}$, and each near 1-factor has precisely $\frac{|V(G)|-1}{2}$ edges, it follows that the near 1-factors in $\mathcal{F}$ are disjoint, and hence $\mathcal{F}$ is the required near 1-factorization. □

4. Some simple classes of graphs

We have already observed that $\chi'_e(K_{2n-1}) = 2n - 1$. In this section we evaluate $\chi'_e(G)$ for some other classes of graphs.

Proposition 10. *Let $H = K(m, n)$ be a complete bipartite graph of odd order, with $m \geq n$. Then $\chi'_e(H) = n+1$ if $H = K(n+1, n)$ and ∞ otherwise.*

Proof. It is easy to see that H has a near 1-factor if and only if $m = n + 1$. Therefore, $\chi'_e(H) = \infty$ if $m \neq n + 1$. Assume now $m = n + 1$. Let G be the graph $K(n + 1, n + 1)$. Since G is 1-factorizable, $\chi'_e(G) = n + 1$. Clearly $H = G - v$. By Proposition 6, $\chi'_e(H) \leq \chi'_e(G) = n + 1$. On the other hand, by Proposition 1, $\chi'_e(H) \geq \Delta(H) = n + 1$. This proves Proposition 10. □

The proofs of the following two propositions are immediate and are left to the reader.

Proposition 11. *Let P_n be a path with n vertices, n odd. Then $\chi'_e(P_n) = 2$.*

Proposition 12. *Let C_n be a cycle with n vertices, n odd. Then $\chi'_e(C_n) = 3$.*

We now prove a result which partially answers the question posed earlier.

Theorem 1. *Let G be a regular graph of even order and let $H = G - v$. Assume $\chi'_e(G) \leq \Delta(G) + 1$. Then $\chi'_e(G) = \chi'_e(H)$.*

Proof. By Proposition 6, H is matching-covered and $\chi'_e(H) \leq \chi'_e(G)$. If G is complete, then the result is obviously true. Otherwise $\Delta(G) = \Delta(H)$. Using the assumption, we have

$$\chi'_e(H) \leq \chi'_e(G) \leq \Delta(G) + 1. \tag{1}$$

Assume first that $\chi'_e(G) = \Delta(G)$. Then

$$\Delta(H) \leq \chi'_e(H) \leq \chi'_e(G) = \Delta(G) = \Delta(H),$$

so that, in particular, $\chi'_e(G) = \chi'_e(H)$.

Assume now $\chi'_e(G) = \Delta(G) + 1$. Since G is regular and of even order, this implies that G is Class 2 (i.e. $\chi'(G) = \Delta(G) + 1$). But then (since G is not complete) also $H = G - v$ is Class 2, which implies that

$$\chi'_e(H) \geq \chi'(H) = \Delta(H) + 1 = \Delta(G) + 1.$$

This, combined with (1), gives the required identity. □

We remark that the statement of Theorem 1 does not hold in general if $\chi'_e(G) = \Delta(G) + 2$, since it is false, e.g., for the Petersen graph.

5. The excessive index of trees

We now consider the class $\mathcal{T}$ of trees. A few definitions will be helpful. A vertex of degree 1 in a tree is called a *leaf* and its unique neighbour is called a *stem*. A *peripheral vertex* of $T \in \mathcal{T}$ is a vertex x such that there exists a vertex y in T such that $dist(x, y) = diam(T)$, where $diam(T)$ is the *diameter* of T (i.e. the length of a longest path).

First we notice the following general fact, which is well known and easy to prove.

Proposition 13. *Let T be a tree of even order. Then T has at most one 1-factor.*

From Proposition 13, $\chi'_e(T)$ can be immediately deduced for all trees T of even order.

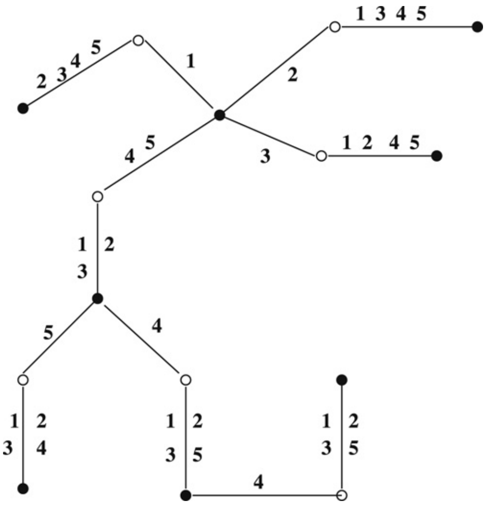

Fig. 2. An example of a matching-covered tree of odd order. The vertices of one of the two colour classes have all degree two. An excessive near 1-factorization is shown.

Corollary 3. *The only tree of even order who has finite excessive index is* K_2.

Proof. Indeed, if T is a tree of order more than two, then it has a vertex v of degree at least two, and, by Proposition 13, at most one of the edges incident with v can be covered by a 1-factor of T. □

Therefore, we may restrict ourselves to trees of odd order. Differently from the even case, there are plenty of trees of odd order which are matching-covered (one of them is depicted in Fig. 2). We need the following definition.

Definition 2. A tree T is a bipolar tree if T has a bipartition (A, B), where all vertices in B have degree two.

Notice that every bipolar tree has necessarily odd order (because it has an even number of edges). An example of bipolar tree is given in Fig. 2.

For convenience we shall call the vertices in B *bivertices*. Notice that the bivertices are precisely those vertices of a bipolar tree which are at odd distance from any leaf.

We now prove that every bipolar tree has an excessive near 1-factorization and we give the exact value of the excessive index.

Theorem 2. *Let T be a bipolar tree and let k be the number of leaves of T. Then T is matching-covered and* $\chi'_e(T) = k$.

Proof. Let $v_1, v_2, \ldots, v_k$ be the leaves of T. For every i, consider the out-tree T_i obtained from T by orienting the edges away from v_i. Let F_i consist of the edges of T corresponding to the arcs of T_i joining each bivertex with its (unique) post-neighbour. Clearly F_i is a near 1-factor of T missing the vertex v_i. Obviously, $\cup_{i=1}^{k} F_i \subset E(T)$. We claim that $\cup_{i=1}^{k} F_i = E(T)$. Let $e \in E(T)$. Then necessarily $e = xy$, where x is a bivertex. It is obvious that there exists a leaf v_i such that $\vec{xy}$ is an arc in T_i. Thus $e \in F_i$, and hence we have the required identity. This proves that T is matching-covered and $\chi'_e(T) \le k$. We now show that $\chi'_e(T) \ge k$. For any leaf v_i, let w_i be the corresponding stem (which is a bivertex). Let u_i be the unique neighbour of w_i different from v_i. Notice that the k edges $e_i = w_i u_i$ are distinct. We show that no two of the e_i's are in the same near 1-factor of T. Assume that there exists a near 1-factor F containing e_i and e_j, where $i \ne j$. Then F must necessarily miss v_i and v_j, which is impossible, since F is a near 1-factor. This proves that the edges $e_1, e_2, \ldots, e_k$ belong to distinct near 1-factors, and hence $\chi'_e(T) \ge k$, concluding the proof of the theorem. □

To complete the classification of trees with respect to the excessive index we now prove that the only matching-covered trees of odd order are the bipolar trees.

Theorem 3. *Let T be a matching-covered tree of odd order. Then T is a bipolar tree.*

Proof. We prove the theorem by induction on the order of T. If $|V(T)| = 1$ the theorem holds trivially, so let us assume that $|V(T)| > 1$. Let v be a peripheral vertex of T and let w be its unique neighbour. Obviously, $\deg(w) \ge 2$ since $T \ne K_2$. Assume $\deg(w) > 2$. Let e_1, e_2 be two distinct edges incident with w and different from the edge vw. Since T is matching-covered, there exists a near 1-factor F_1 containing the edge e_1 and a near 1-factor F_2 containing the edge e_2. Obviously, $F_1 \ne F_2$ because the edges e_1, e_2 are adjacent. Both F_1 and F_2 do not contain the edge vw, and hence they miss the vertex v. But then F_1 and F_2 are 1-factors of the tree $T_1 = T - v$, and hence they must coincide by Proposition 13. This contradiction shows that $\deg_T(w) = 2$. Let u be the unique neighbour of w different from v. Consider the tree $T_2 = T - \{v, w\}$. Clearly T_2 has odd order.

Claim 1. *T_2 is matching-covered.*

Let $e \in E(T_2)$. Since T is matching-covered, there exists a near 1-factor F of T containing e. Obviously, F contains either the edge vw or the edge wu. In either case the matching $F - vw$ (or $F - wu$) is a near 1-factor of T_2 containing the edge e. Thus T_2 is matching-covered.

Claim 2. *T_2 is a bipolar tree and u is not a bivertex of T_2.*

The fact that T_2 is a bipolar tree follows from the fact that T_2 is matching-covered, $|V(T_2)| < |V(T)|$ and the inductive hypothesis. We now prove that u is not a bivertex of T_2. We argue by contradiction, so let us assume that u is a bivertex of T_2. Let F be a near 1-factor of T containing the edge uw. Then $F_2 = F - uw$ is a near 1-factor of T_2 missing the vertex u. But then the graph $H = T_2 - u$ must have a 1-factor. However, it is easily seen that H is the union of bipolar trees and, since all bipolar trees have odd order, in particular H cannot have a 1-factor. This contradiction proves the claim.

Now, by Claim 2, T_2 is a bipolar tree and u is not a bivertex of T_2. It follows that T is a bipolar tree. By induction, the proof is completed. □

Thus, we are now in a position to express the excessive index of an arbitrary tree as follows.

Corollary 4. *Let T be a tree. Then*

$$\chi'_e(T) = \begin{cases} 1 & \text{if } T = K_2; \\ \#\text{ leaves} & \text{if } T \text{ is a bipolar tree;} \\ \infty & \text{otherwise.} \end{cases}$$

6. Conclusion

For any class of graphs not considered here, one may ask: what is the excessive index of such graphs? The authors have, for instance, investigated the excessive index of the complete multipartite graphs of even order [2]. It was not easy to come up with the following solution.

Theorem 4 ([2]). *Let $G = K(n_1, n_2, \ldots, n_{r-1}, n_r)$ be a complete multipartite graph of even order, where $n_1 \geq n_2 \geq \cdots \geq n_{r-1} \geq n_r$ are the sizes of the partite sets and $r \geq 3$. Then $\chi'_e(G) < \infty$ if and only if $n_1 < \sum_{i=2}^{r} n_i$, in which case $\chi'_e(G) = \max\{\Delta(G), \sigma_1(G)\}$, where*

$$\sigma_1(G) = \lceil |E(K(n_2, n_3, \ldots, n_r))| / (|V(G)| - 2n_1) \rceil .$$

The task of determining the excessive index of complete multipartite graphs of *odd* order appears to the authors to be an intriguing, but formidable one.

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