On the Diameter of the Generalized Undirected de Bruijn Graphs $UG_B(n, m)$, $n^2 < m \leq n^3$

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The generalized de Bruijn digraph $G_B(n, m)$ is the digraph (V, A) where $V = \{0, 1, \ldots, m-1\}$ and $(i, j) \in A$ if and only if $j \equiv in + \alpha \pmod{m}$ for some $\alpha \in \{0, 1, 2, \ldots, n-1\}$. By replacing each arc of $G_B(n, m)$ with an undirected edge and eliminating loops and multi-edges, we obtain the generalized undirected de Bruijn graph $UG_B(n, m)$. In this article, we prove that when $2n^2 \leq m \leq n^3$ the diameter of $UG_B(n, m)$ is equal to 3. We also show that for pairs (n, m) where $n^2 < m < 2n^2$ the diameter of $UG_B(n, m)$ can be 2 or 3. © 2008 Wiley Periodicals, Inc. NETWORKS, Vol. 52(4), 180–182 2008

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1. INTRODUCTION

All graphs considered in this study are undirected, loopless, and without multi-edges. For graph theory terminology, we follow [9]. For brevity, $[a, b] = \{a, a+1, \ldots, b\}$ is defined here for non-negative integers a and b where $a < b$.

Imase and Itoh [6] were the first to generalize the well-known de Bruijn network $B(d, n)$, independently followed by Reddy et al. [8]. The generalized de Bruijn digraph $G_B(n, m)$ is the directed graph, whose vertices are $0, 1, \ldots, m-1$, and whose directed edges (arcs) are of the form

$$i \rightarrow in + \alpha \pmod{m}, \quad \forall i \in [0, m-1] \text{ and } \forall \alpha \in [0, n-1].$$

The generalized undirected de Bruijn graph is the undirected graph derived from the generalized de Bruijn digraph by replacing directed edges with undirected edges and omitting loops and multi-edges. Such a graph is denoted here by $UG_B(n, m)$. The set of neighbors of any vertex i in $UG_B(n, m)$ is $N(i) = R(i) \cup L(i)$, where

$$R(i) = \{in + \alpha \pmod{m} : \alpha \in [0, n-1]\} \quad \text{and}$$
$$L(i) = \{j : jn + \beta \equiv i \pmod{m},$$
$$\text{where } \beta \in [0, n-1], j \in [0, m-1]\}.$$

Therefore, if $j \in R(i)$ then $i \in L(j)$.

Imase and Itoh [6] proved that the generalized de Bruijn digraph $G_B(n, m)$ is $(n-1)$-connected and its diameter is bounded above by $\lceil \log_n m \rceil$. Therefore, $UG_B(n, m)$ possesses the same properties.

In the study of fault tolerance and transmission delay of networks, the connectivity and diameter of the graph are two very important parameters; these have been thoroughly studied by many authors [3, 4, 6, 10]. Since the de Bruijn graphs $B(d, n)$ are known to have short diameters and simple routing strategies, they have been widely used as models for communication networks and multiprocessor systems [4]. However, one of the disadvantages of de Bruijn graphs $B(d, n)$ is the restriction on the number of vertices d^n. The generalized de Bruijn graphs retain all of the properties of the de Bruijn graphs, but have no restrictions on the number of vertices [4]. So, determining the connectivity and diameter of $UG_B(n, m)$ is of relevant interest and importance.

Caro et al. [2] proved that $UG_B(n, n(n+1))$ has a w-wide-diameter of 5 for $w = 2(n-1)$. Escuadro and Muga [5] proved that $UG_B(n, n^2)$ is $2(n-1)$-regular and has diameter 2; in addition, they showed that the w-wide-diameter of $UG_B(n, n^2)$ is 4 for $w = 2(n-1)$ and $n \geq 4$. Nochefranca and Sy [7] proved that the diameter of $UG_B(n, n(n^2+1))$ is 4 for odd $n \geq 3$. Caro and Zeratsion [3] recently proved that the diameter of $UG_B(n, m)$ is 2 for $m \in [n+1, n^2]$, and 3 for $m \in [n^2+1, n^3]$ where n divides m. Caro et al. [1] also provided an upper bound for the diameter of $UG_B(n, n^2+1)$ when $n \geq 5$ is odd.

This work shows that the diameter of $UG_B(n, m)$ is 3 whenever $n \geq 2$ and $2n^2 \leq m \leq n^3$. Notably, for $n^2 < m < 2n^2$, there are pairs (n, m) such that the diameter of $UG_B(n, m)$ is 2 or 3. This work also verifies that the diameter of $UG_B(n, n^2+1)$ is 3 and the diameter of $UG_B(n, n^2+2)$ is 2.

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2. $UG_B(n,m)$, $2n^2 \le m \le n^3$

Let $d_G(x,y)$ denote the distance between two vertices x and y in a graph (or directed graph) G, and let $d(G)$ denote the diameter of the graph G. We use $< u, \ldots, v >$ to denote a path from u to v in G.

Imase and Itoh [6] proved that the diameter of the generalized de Bruijn digraph $G_B(n,m)$ is bounded above by $\lceil \log_n m \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x. Since for any two distinct vertices u and v in $UG_B(n,m)$, the distance from u to v in the corresponding $G_B(n,m)$ provides an upper bound for the distance between u and v, we have $d(UG_B(n,m)) \le d(G_B(n,m))$. Therefore, the following bound is immediate.

Lemma 2.1. *The diameter of the generalized undirected de Bruijn graph $UG_B(n,m)$ is at most $\lceil \log_n m \rceil$.*

On the other hand, in $UG_B(n,m)$, the degree of every vertex is at most $2n$. Therefore, from a vertex u of degree $2n-1$, one can reach at most $(2n-1) + (2n-1)^2 + \cdots + (2n-1)^d$ vertices via a path of length d. With this observation, we get the following lower bound for the diameter.

Lemma 2.2. $\lceil \log_{2n-1} m \rceil \le d(UG_B(n,m))$ *for $n+1 \le m$.*

Corollary 2.3. *The diameter of $UG_B(n,m)$ is 2 or 3 for $n^2 \le m \le n^3$.*

Proof. By Lemma 2.1 and $\lceil \log_{2n-1} n^2 \rceil = 2$ for $n > 2$, we have the result. ∎

Now, we are ready to show our main results.

Theorem 2.4. *For positive integers $n \ge 2$ and $2n^2 \le m \le n^3$, the diameter of $UG_B(n,m)$ is 3.*

Proof. Let $[0, m-1]$ be the vertex set of $G = UG_B(n,m)$. We claim that either $d_G(0, m-n) = 3$ or $d_G(0, m-n-1) = 3$. For convenience, let $j_1 = m-n$ and $j_2 = m-n-1$. By inspection, we have $j_1 \notin N(0)$ and $j_2 \notin N(0)$. Therefore, it suffices to prove that either $N(0) \cap N(j_1) = \emptyset$ or $N(0) \cap N(j_2) = \emptyset$, which implies that $d(G) \ge 3$. Then, by Corollary 2.3, the result follows.

By definition, $N(0) = R(0) \cup L(0)$ and $N(j) = R(j) \cup L(j)$ where $j = j_1$ or j_2 as the case may be. Therefore, it is equivalent to show that $[R(0) \cup L(0)] \cap [R(j) \cup L(j)] = \emptyset$. We split the proof into four cases with the first three cases dealing with $j = j_1$ or j_2.

Case 1. $R(0) \cap L(j) = \emptyset$. Since $\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1,n-1]} R(i) = [n, n^2-1]$, neither j_1 nor j_2 are in $\bigcup_{i \in R(0)} R(i)$. This implies that $R(0) \cap L(j) = \emptyset$.

Case 2. $R(0) \cap R(j) = \emptyset$. By the definition of $R(j)$, $R(j) = \{jn + \alpha \pmod m : \alpha \in [0, n-1]\}$. Hence, it is clear that $R(0) \cap R(j) = \emptyset$.

Case 3. $L(0) \cap L(j) = \emptyset$. Assume that $L(0) \cap L(j) \ne \emptyset$. Then there exists a k such that $0 \in R(k)$ and $j \in R(k)$. This implies that there exist α and β where $0 \le \alpha, \beta \le n-1$ satisfying

$$\begin{cases} kn + \alpha \equiv 0 \pmod m, \\ kn + \beta \equiv j \pmod m. \end{cases} \tag{2.1}$$

Therefore, $\beta - \alpha \equiv j \pmod m$ and $-(n-1) \le \beta - \alpha \le n-1$. Since $\beta - \alpha \ne j$ if $\beta - \alpha \ge 0$ and $(-\beta + \alpha) + m - n < m$ or $(-\beta + \alpha) + m - n - 1 < m$, we conclude that no solution (α, β) exists for (2.1). Hence the case is proved.

Case 4. $L(0) \cap R(j) = \emptyset$, $j = j_1$ or j_2. First, we define $\delta(j_1) = 0$ and $\delta(j_2) = 1$. We claim that either $0 \notin \bigcup_{i \in R(j_1)} R(i)$ or $0 \notin \bigcup_{i \in R(j_2)} R(i)$. Assume that the above assertion is not true. Then, there exist $0 \le \alpha, \beta, \gamma, \epsilon \le n-1$ such that

$$\begin{cases} ((m - n - \delta(j_1))n + \alpha)n + \beta \equiv 0 \pmod m, \\ ((m - n - \delta(j_2))n + \gamma)n + \epsilon \equiv 0 \pmod m. \end{cases}$$

Thus,

$$\begin{cases} -n^3 + \alpha n + \beta \equiv 0 \pmod m, \\ -n^3 - n^2 + \gamma n + \epsilon \equiv 0 \pmod m. \end{cases}$$

This implies that $n^2 + (\alpha - \gamma)n + (\beta - \epsilon) \equiv 0 \pmod m$. Since both $\alpha - \gamma$ and $\beta - \epsilon$ are integers between $-(n-1)$ and $(n-1)$, we have $2n^2 > n^2 + (\alpha - \gamma)n + (\beta - \epsilon) > 0$. Therefore, we are not able to find $(\alpha, \beta, \gamma, \epsilon)$ to satisfy $n^2 + (\alpha - \gamma)n + (\beta - \epsilon) \equiv 0 \pmod m$. Hence, we conclude that either $0 \notin \bigcup_{i \in R(j_1)} R(i)$ or $0 \notin \bigcup_{i \in R(j_2)} R(i)$ and thus either $L(0) \cap R(j_1) = \emptyset$ or $L(0) \cap R(j_2) = \emptyset$.

Now, combining the above four cases and $j \notin N(0)$, we have either $d_G(0, j_1) = 3$ or $d_G(0, j_2) = 3$. ∎

3. $UG_B(n,m)$, $n^2 < m < 2n^2$

Similar to Theorem 2.4, if we can find two vertices $i, j \in [0, m-1]$ such that $d_G(i,j) \ge 3$, then we can show that $d(G) \ge 3$. First, we find the diameter of $UG_B(n, n^2+1)$.

Proposition 3.1. $d(UG_B(n, n^2+1)) = 3$ *for $n \ge 4$.*

Proof. Let $m = n^2+1$ and $n \ge 4$. Consider $i = n-2$ and $j = n^2 - n + 2$ in $G = UG_B(n,m)$. We claim $d_G(i,j) \ge 3$. Since $(n^2 - n + 2)n + \alpha \equiv n + 1 + \alpha \pmod m > n - 2 = i$ and $(n-2)n + \alpha \le n^2 - n - 1 < n^2 - n + 2 = j$, $i \notin R(j)$ and $j \notin R(i)$ follow. Hence, it suffices to show that $[R(i) \cup L(i)] \cap [R(j) \cup L(j)] = \emptyset$ which can be broken down into four cases.

- $R(i) \cap R(j) = \emptyset$
  Since $(n-2)n + \alpha \equiv (n^2 - n + 2)n + \beta \pmod m$, $\alpha - \beta \equiv 3n + 2 \pmod m$. Clearly, there are no solutions for α and β when $n \ge 4$.
- $R(i) \cap L(j) = \emptyset$
  Since $(ni + \alpha)n + \beta \equiv j \pmod m$, we have $\alpha n + \beta \equiv i + j = n^2 \pmod m$. By the fact $|\alpha n + \beta| \le n^2 - 1$, there are no solutions for α and β.

- $L(i) \cap R(j) = \emptyset$
  Since $(nj + \alpha)n + \beta \equiv i \pmod{m}$, we have $\alpha n + \beta \equiv n^2 \pmod{m}$ and we are not able to find solutions for α and β.
- $L(i) \cap L(j) = \emptyset$
  Suppose not. Then there must exist $k \in [0, n^2]$ satisfying $kn + \alpha \equiv i \pmod{m}$ and $kn + \beta \equiv j \pmod{m}$. Therefore, $|\alpha - \beta| = |i - j| = |n^2 - 2n + 4| > n - 1$. Again, this is not possible.

We note here that $d(UG_B(n, n^2 + 1)) = 2$ for $n = 2, 3$.

To show the diameter of $UG_B(n, m)$ is equal to 2 for some $n^2 < m < 2n^2$, we have to make sure that for each pair of vertices i and j, $N(i) \cap N(j) \neq \emptyset$ or $i \in N(j)$. Surprisingly, if $m = n^2 + 2$, then the diameter of $UG_B(n, m)$ is equal to 2.

Proposition 3.2. $d(UG_B(n, n^2 + 2)) = 2$ *for* $n \geq 3$.

Proof. Let $m = n^2 + 2$. For any two distinct vertices x and y in $UG_B(n, m)$, we claim that $d_G(x, y) \leq 2$. It suffices to show that $N(x) \cap N(y) \neq \emptyset$. Since $N(x) = R(x) \cup L(x)$ and $N(y) = R(y) \cup L(y)$, we have to prove that one of the following four conditions holds: (1) $R(x) \cap L(y) \neq \emptyset$, (2) $R(y) \cap L(x) \neq \emptyset$, (3) $R(x) \cap R(y) \neq \emptyset$ or (4) $L(x) \cap L(y) \neq \emptyset$.

Observe that $R(x) \cap L(y) \neq \emptyset$ if and only if $(nx + \alpha)n + \beta \equiv y \pmod{m}$ for some $0 \leq \alpha, \beta \leq n - 1$. Therefore, $y + 2x \equiv \alpha n + \beta \in [0, n^2 - 1] \pmod{m}$. In fact, $\{\alpha n + \beta : 0 \leq \alpha, \beta \leq n - 1\} = [0, n^2 - 1]$. This implies that if $y + 2x \in [0, n^2 - 1] \pmod{m}$, then $d(x, y) \leq 2$. On the other hand, by considering $R(y) \cap L(x) \neq \emptyset$, we have that if $x + 2y \in [0, n^2 - 1] \pmod{m}$, then $d(x, y) \leq 2$.

So, assume $x + 2y$ and $2x + y$ are equal to either n^2 or $n^2 + 1 \pmod{m}$. Since $0 \leq x \neq y \leq n^2 + 1$, there are only six possible cases to consider.

But, if $2x + y = n^2$ and $2y + x = 2n^2 + 2$, then $3n^2 + 2 \equiv 0 \pmod{3}$ which is not possible. By the same reason, $2x + y = n^2 + 1$ and $2y + x = 2n^2 + 3$ are not possible. Furthermore, if $2x + y = n^2$ and $2y + x = 2n^2 + 3$, then $y - x = n^2 + 3$, which is not possible, either. Thus, we have exactly three cases to check.

- $2x + y = n^2$ and $2y + x = n^2 + 1$
  In this case, since $2n^2 + 1 \equiv 0 \pmod{3}$, we may let $n = 3p + 1$. Then $x = 3p^2 + 2p$ and $y = 3p^2 + 2p + 1$. Hence, we have a path $< 3p^2 + 2p, p, 3p^2 + 2p + 1 >$ from x to y, which concludes the proof.
- $2x + y = n^2 + 1$ and $2y + x = 2n^2 + 2$
  We have $x = 0$ and $y = n^2 + 1$. Therefore, the path $< 0, n, n^2 + 1 >$ connects x and y for $n \geq 3$, giving the result.
- $2x + y = 2n^2 + 2$ and $2y + x = 2n^2 + 3$
  Since $4n^2 + 5 \equiv 0 \pmod{3}$, it suffices to consider the cases $n \equiv 1, 2 \pmod{3}$. First, if $n = 3p + 1$, then let $x = 6p^2 + 4p + 2$ and $y = 6p^2 + 4p + 1$. It is easy to see that $< 6p^2 + 4p + 1, 2p, 6p^2 + 4p + 2 >$ is a path from x to y. If $n = 3p + 2$, the proof follows by letting $x = 6p^2 + 8p + 4$ and $y = 6p^2 + 8p + 3$.

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