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MORE RESULTS ON THE ORTHOGONAL LATIN SQUARE GRAPHS

Hung-Lin Fu

Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan
REPUBLIC OF CHINA

1. Introduction.

Two $n \times n$ Latin squares are orthogonal provided that when they are superimposed the ordered pairs obtained are all distinct. An orthogonal Latin square graph (OLSG) is one in which the vertices are distinct Latin squares of the same order and on the same symbols, and two vertices are adjacent if and only if the Latin squares are orthogonal. If G is an arbitrary (finite) graph, we say that G is realizable as an OLSG if there is an OLSG isomorphic to G. The spectrum of G, Spec(G), is defined as the set of all positive integers n such that there is a realization of G by Latin squares of order n.

In [5], it has been shown that every graph is realizable and Spec(G) contains all but a finite set of integers, that is, there exists an integer v_0, such that for every $v \geq v_0$, $v \in$ Spec(G). But, no explicit bounds on v_0 are given. In [3], a better bound v_0 has been given in the case G is a bipartite graph. In this paper, we will give an explicit bound v_0 for the graph $K_3 \cup \{e\}$ mentioned in [5], and some other graphs.

2. The main theorems.

Let L and M be a pair of orthogonal Latin squares based on the set $S = \{1,2,\ldots,n\}$ and α, β be permutations on S. It is well-known that L_α and M_β are orthogonal, where L_α and M_β are the Latin squares obtained from permuting the entries of L and M by α and β, respectively. (In what follows, the notation for the graphs are mainly from [4].) Also, if L contains a subsquare C of order m, we can unplug C and plug in another Latin square C', denoted by $(L \backslash C) \cup C'$.

Let G be a disjoint union of two graphs G_1 and G_2. It is natural to ask the following question about spectrum. If $v \in$ Spec$(G_1) \cap$ Spec(G_2), can we conclude that $v \in$ Spec(G) whenever v is large enough. The answer is "yes". Before we prove the theorem, we need two lemmas.

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Lemma 2.1. *Let L and M be two orthogonal Latin squares of order n, and L' be a Latin square obtained by permuting two columns of L, then L' and M are not orthogonal.*

Proof: The proof followed by looking at the special entry "a" of those two columns being permuted, the ordered pairs (a, b) and (a, c) occurred when we superimposed L and M are no longer in the set of ordered pairs when we superimpose L' and M. ∎

Lemma 2.2. *If L and M are two Latin squares of order $v \geq 4$ which are not orthogonal, but L' is orthogonal to M where L' is obtained by permuting two columns of L, then there is a transposition of columns of L such that after the transposition the Latin square is not orthogonal to M.*

Proof: Let $L = [l_{i,j}]$, $M = [m_{i,j}]$ be two Latin squares of the order v which are not orthogonal, and denote the Latin square L' obtained from permuting columns a and b by $L_{(a,b)}$. Since L and M are not orthogonal, there exists a pair of entries, (i, j) and (i', j') such that $l_{i,j} = l_{i',j'}$ and $m_{i,j} = m_{i',j'}$. It is not difficult to see the intersection of $\{a, b\}$ and $\{j, j'\}$ must be nonempty. (L' and M are orthogonal.) Hence, if we pick the transposition (cd) such that $\{c, d\} \cap \{j, j'\} = \phi$ then $L_{(c,d)}$ is not orthogonal to M. We conclude the proof. ∎

Theorem 2.3. *Let G be a graph with two disjoint subgraphs G_1 and G_2. If $v \in Spec(G_1) \cap Spec(G_2)$ and $|V(G_1)| \cdot |V(G_2)| \leq \frac{v}{2}$ then $v \in Spec(G)$.*

Proof: Let G_1, G_2 be two OLSG realized by distinct Latin squares of order v such that $|V(G_1)| \cdot |V(G_2)| \leq \frac{v}{2}$. If every Latin square in G_1 is not orthogonal to any Latin square in G_2, we are done. Otherwise, we can apply Lemma 2.1, and Lemma 2.2, to find a transposition of columns of all Latin squares in G_1 such that no Latin square in G_1 (new) is orthogonal to any Latin square in G_2. (We consider all the Latin squares in G_1, such that the Latin square is not orthogonal to any Latin square in G_2, but after permuting columns with a transposition, the new Latin square is orthogonal to a Latin square in G_2.) From the proof of Lemma 2.2, we notice that the choice of (cd) is independent to the choice of (ab), but depend on $\{j, j'\}$. Since there are at most $|V(G_1)| \cdot |V(G_2)| - 1$ pairs of $\{j, j'\}$, hence, we can always find (cd) if $v \geq 2$ $|V(G_1)| \cdot |V(G_2)|$. This concludes the proof. ∎

As an example of the above theorem, the spectrum of all the disconnected graphs of order not greater than 4 can be computed easily.

In [3], the idea of generalized direct product of Latin squares and the embedding of a pair of orthogonal Latin squares to two mutually orthogonal Latin squares have been used to give an explicit bound for bipartite graphs. We can apply a similar idea to find the explicit bound of n-partite graphs.

Let B be a $p \times p$ Latin square, and for each ordered pair (i,j), $i,j \in \{1,2,\dots,p\}$, let A_j^i be a $q \times q$ Latin square. Then the generalized direct product of the A_j^is and B (denoted by $A_j^i \times B$ or $A \times B$ if all of the A_j^is = A) is given by the accompanying diagram (Figure 2.1, where $A_j^i \times B(i,j)$ is the Latin square obtained from A_j^i by replacing each entry a in A_j^i by $(a, B(i,j))$, where $B(i,j)$ is the entry in cell (i,j) of B.

$A_1^1 \times B(1,1)$	$A_2^1 \times B(1,2)$	$\dots$	$A_p^1 \times B(1,p)$
$A_1^2 \times B(2,1)$	$A_2^2 \times B(2,2)$	$\dots$	$A_p^2 \times B(2,p)$
$\vdots$	$\vdots$		
$A_1^p \times B(p,1)$	$A_2^p \times B(p,2)$	$\dots$	$A_p^p \times B(p,p)$

Figure 2.1

It is a routine matter to check if B and $\overline{B}$ are orthogonal Latin squares of order p, and for each ordered pair (i,j), A_j^i is orthogonal to $\overline{A}_j^i$, then $A_j^i \times B$ and $\overline{A}_j^i \times \overline{B}$ are orthogonal Latin squares of order $p \cdot q$.

Theorem 2.4. *If there exist n mutually orthogonal Latin squares of order p, and $n+1$ mutually orthogonal Latin squares of order q, then $p \cdot q \in Spec(G)$ where G is an n-partite graph such that $|V(G)| \leq p^2$.*

Proof: Let $L_1, L_2, \dots, L_n$ be n mutually orthogonal Latin squares of order p, and $M_1, M_2, \dots, M_{n+1}$ be $n+1$ mutually orthogonal Latin squares of order q. Let $G = (H_1, H_2, \dots, H_n)$, and the vertices in H_t are $A_j^i \times L_t$ where $A_j^i = (M_t)_\alpha$, α is a permutation on $\{1,2,\dots,q\}$. It is not difficult to see that we start with a complete n-partite graph. Since $|V(G)| \leq p^2$, for each vertex in G we can choose a corresponding block from those p^2 blocks in Figure 2.1. (Two different vertices should have two distinct corresponding blocks.) Now, if there is any edge missing between H_s and H_t, $s < t$, we can replace the block corresponding to the vertex in H_s which is $A_j^i = (M_s)_\beta$, by $A_j^i \times L_s$, and the corresponding block in the other vertex is also replaced by the same $A_j^i \times L_s$, where $A_j^i = (M_{n+1})_\gamma$ and fix all the other blocks. (β and γ are permutations on $\{1,2,\dots,q\}$). We continue this process until we take care of the all missing edges, this concludes the proof. (We can, of course, pick suitable permutation, in order that all the Latin squares are distinct.) ∎

A quick result from above, $7 \cdot 127 \in$ Spec(G) if G is a 5-partite graph, and $|V(G)| \leq 127^2$.

Lemma 2.5. *If n mutually orthogonal Latin squares of order u can be embedded in n mutually orthogonal Latin squares of order v, and $u \in Spec(G)$ where G is an n-partite graph, then $v \in Spec(G)$.*

Proof: The proof followed by embedding every Latin square of order u in each part to a Latin square of those n mutually orthogonal Latin squares of order v.
∎

With the above lemma, we can find an explicit bound for an n-partite graph, if we have the information about embedding n mutually orthogonal Latin squares. As an example, since 5 mutually orthogonal Latin squares of order $k \geq 837$ can be embedded in 5 mutually orthogonal Latin squares of order $v \geq 7k + 7$ [2], hence, we have the following corollary.

Corollary 2.6. *If G is a 5-partite graph with $|V(G)| \leq (127)^2$, then $v \in Spec(G)$ for every $v \geq 7 \cdot 7 \cdot 127 \cdot +7 = 6230$.*

Proof: Since $7 \cdot 127 \in \mathrm{Spec}(G)$, hence, we have the proof. ∎

Corollary 2.7. *If G is a 5-partite graph, then $v \in Spec(G)$ for every $v \geq 7 \cdot 2^{n+3} + 7$ where $2^{2n} \geq |V(G)|$, and $n \geq 7$.*

Proof: It is well known that there exist $2^n - 1$ mutually orthogonal Latin squares of order 2^n, hence, by Lemma 2.4, Lemma 2.5 and [2], we have the proof. ∎

3. OLSG of small order.

For the OLSG of smaller order we have a better bound which we will discuss below.

Before we go any further, we need the following notations and a lemma [6]. We let $H = N \setminus \{1, 2, 6\}$ (N is the set of all positive integers), $I_t = \{v: v \in N$ and $v \geq t\}$, $H_3 \subseteq \{30, 34, 38, 41, 44, 45, 48\}$, $E_3 \subseteq \{35, 46\}$,
$N_3 \subseteq \{38, 42, 50\}$, $H_4 \subseteq \{37, 38, \ldots, 48, 52, 53, 60, 74\}$,
$E_4 \subseteq \{41, 42, \ldots, 48, 52, 53, 60, 74\}$,
$N_4 \subseteq \{47, 48, 49, 50, 52, 53, 54, 56, 60, 74\}$,
$H_5 \subseteq \{42, \ldots, 48, 51, 52, \ldots, 55, 59, 60, 61, 62, 67, 68, 74, 75, 76, 83\}$,
$E_5 \subseteq \{48, 51, 52, \ldots, 55, 59, 60, 61, 62, 67, 68, 69, 74, 76, 83\}$,
$N_5 \subseteq \{54, 55, 56, 59, 60, 61, 62, 67, 68, 69, 74, 76, 83, 84\}$.

Lemma 3.1 [6]. *Let $P_k(n)$ be the set of all positive integers v such that there exists k mutually orthogonal Latin squares of order v which contains k mutually orthogonal Latin subsquares of order n, then we have (i) $P_3(7) = I_{28} \setminus H_3$, (ii) $P_3(8) = I_{32} \setminus E_3$, (iii) $P_3(9) = I_{36} \setminus N_3$, (iv) $P_4(7) = I_{35} \setminus H_4$, (v) $P_4(8) = I_{40} \setminus E_4$, (vi) $P_4(9) = I_{45} \setminus N_4$, (vii) $P_5(7) = I_{42} \setminus H_5$, (viii) $P_5(8) = I_{48} \setminus E_5$, (ix) $P_5(9) = I_{54} \setminus N_5$.*

Theorem 3.2. *Let G be the graph in Figure 3.2, then $v \in Spec(G)$ if $v \geq 31$.*

Proof: Since (by Lemma 3.1) there are three mutually orthogonal Latin squares of order $u(= 7, 8, \text{or } 9)$ which can be embedded in three mutually orthogonal Latin squares of order $v \geq 31$, hence, we have the proof by assigning Latin squares to the following graphs. (Let C_1, C_2, C_3 be the three mutually orthogonal Latin squares of order u, and L_1, L_2, L_3 be the three mutually orthogonal Latin squares of order v.) ∎

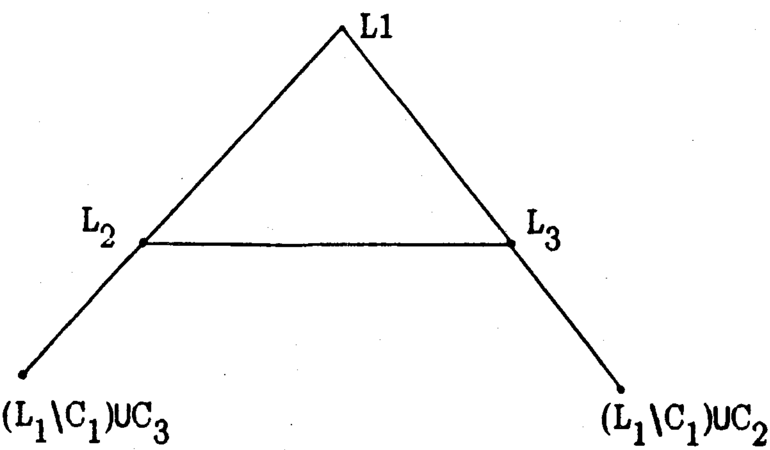

Figure 3.2

The above theorem answers the question mentioned in [5]. There are some other graphs which can be realized by using the technique in the proof of Theorem 3.2, we omit the detail.

One more result is worth mention.

Theorem 3.3. *Let G be the graphs $K_5\backslash K_2$, $K_5\backslash K_3$, $K_5\backslash(K_2 \cup K_2)$, or $K_5\backslash(K_3 \cup K_2)$, then $v \in Spec(G)$ if $v \geq 84$.*

Proof: By Lemma 3.1, there ae five mutually orthogonal Latin squares of order $v \geq 84$ which contains 5 mutually orthogonal Latin squares of order 8, hence, we can replace the subsquares by two extra orthogonal Latin squares suitably to obtain the above OLSG. ∎

Remark: We wish some better ideas can be found in order to give an explicit bound for all graphs. Of course, we would like to have this bound as small as possible.

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