New Results on Harmonious Trees

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Abstract. In this paper, we mainly obtain a couple of new constructions for constructing harmonious trees from smaller sequential trees. Subsequently, we obtain two new classes of harmonious trees called palm tree and firework. As a consequence of adjoining an even number of paths, we disprove an earlier claim by Cahit [1] in investigating harmonious p-stars.

1. Introduction

For a finite simple graph G with p vertices and q edges, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. A *harmonious labeling* of a connected graph G is an injection $g: V(G) \to \mathbf{Z}_q$ if the induced labeling on edge set $g^*: E(G) \to \mathbf{Z}_q$ defined by $g^*(uv) \equiv g(u) + g(v) \pmod q$ for each edge $uv \in E(G)$ is 1-1 when G is not a tree. If G is a tree, then either exactly one vertex label is allowed to repeat or the vertices can be labeled by using $\{0, 1, 2, \ldots, q\}$. In both cases, the resulting edge labels are distinct after taking modulo q. For clarity, we call the former case a *harmonious labeling of type* 1 and the latter case a *harmonious labeling of type* 2. Either type of labeling for a tree is a harmonious labeling. A graph G with a harmonious labeling is called a *harmonious graph.*

Harmonious labeling on graphs was introduced in 1980 by Graham and Sloane [5] which is related to a problem of modular versions of certain additive bases for the integers. Results on the existence of known harmonious graphs can

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be found in a survey by Gallian [3].

In order to construct new harmonious graphs we shall make use of a stronger version of harmonious graphs. A vertex labeling of G, $f : V(G) \rightarrow \{0,1,...,q-1\}$, is called *strongly k-harmonious* [2] if the edge labels induced by $f^{\#}(uv) = f(u)+f(v)$ for each edge $uv \in E(G)$ form a sequence of consecutive integers k, $k+1$, $k+2$, ..., $k+q-1$. Since the edge labels are sequential, a strongly k-harmonious labeling is also known as a *k-sequential* labeling. Similarly, when G is a tree, it is called a *k-sequential labeling of type* 1 if its vertices are labeled from 0 to $q-1$ with one repeated label and a *k-sequential labeling of type* 2 if its vertices are labeled from 0 to q with no repeated labels. Both types are k-sequential labelings. A graph G is a *sequential* graph [4] if G has a k-sequential labeling for some positive integer k. Note that a sequential graph is clearly a harmonious graph. Results on sequential graphs can also be found in [3].

It is worth of mentioning that a sequential labeling of type 2 does have a better property. Assume that g is a k-sequential labeling of type 2 of a tree G with p vertices. Then, the labeling $g^{c}(u) = p-1-g(u)$ for each $u \in V(G)$ is a $(p-k)$-sequential labeling of type 2 of G. Therefore, G is both k and $(p-k)$-sequential. The labeling g^{c} is called a *complementary labeling* of g.

In this paper, we shall use a couple of graph operations called "attaching and adjoining" to cluster sequential graphs together and obtain new classes of sequential graphs or harmonious graphs as the case may be. For example, the p-stars studied by Cahit in [1] can be obtained by adjoining paths to a common vertex.

2. Attaching construction

We start this section with the definition of attaching operation. Let G_1, G_2,..., G_n be a collection of vertex-disjoint graphs and H be a graph with n vertices. Let $V(H)$ = $\{w_1, w_2, ..., w_n\}$ and let $v_i \in V(G_i)$, for i= 1, 2,..., n. Then, by identifying w_i and v_i for each i = 1, 2,..., n, we obtain a graph which attaches G_1, G_2,..., G_n to H at $w_1, w_2, ..., w_n$ respectively. Denote the graph obtained above by $H \oplus [G_1, G_2, ..., G_n]$ at $[v_1, v_2, ..., v_n]$ (Depicted in Figure 1.). In particular, if $G_i \cong G$ for i = 1, 2, ..., n and $v_1, v_2, ..., v_n$ are corresponding to the same vertex v in $V(G)$ (under isomorphism), then the graph is denoted by $H \oplus [G]_v$. Furthermore, we use $H \oplus [G]$ to denote the collection of $|V(G)|$ graphs $H \oplus [G]_v$ where $v \in V(G)$.

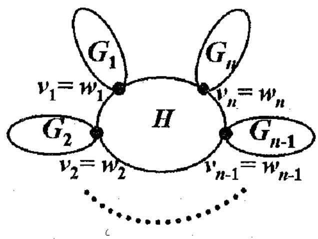


Figure 1. $H \oplus [G_1, G_2, \ldots, G_n]$ at $[v_1, v_2, \ldots, v_n]$

In what follows, we shall mainly consider the graphs $H \oplus [G]$ where H is of order n and there are n isomorphic copies of G with p vertices. For convenience, throughout this paper, we let $V(G) = \{u_k \mid k \in \mathbf{Z}_p\}$ and the ith copy of G be denoted by G_i and $V(G_i) = \{ u_{i,j} \mid j \in \mathbf{Z}_p \}$ where $u_{i,k}$ is the isomorphic image of u_k, $k \in \mathbf{Z}_p$. In this section, the sequential labelings of trees refer to type 2.

Theorem 2.1. *(Attaching construction) Suppose that H is an r-sequential tree with* $2n+1$ *vertices and G is a tree with p vertices which has a k-sequential labeling g.*

Then we have the following:

(a) *If* $r = n+1$, *k is odd, and* $g(u_0) = (k-1)/2$, *then* $H \oplus [G]_{u_0}$ *is* $(k(2n+1)-n)$*-sequential.*

(b) *If* $r = n+1$, *k is even and p is odd, and* $g(u_0) = (p+k-1)/2$, *then* $H \oplus [G]_{u_0}$ *is* $((p-k)(2n+1)-n)$*-sequential.*

(c) *If* $r = n$, $k+p$ *is odd, and* $g(u_0) = (p+k-1)/2$, *then* $H \oplus [G]_{u_0}$ *is* $(k(2n+1)+n)$*-sequential.*

(d) *If* $r = n$, *kp is odd and* $g(u_0) = (k-1)/2$, *then* $H \oplus [G]_{u_0}$ *is* $((p-k)(2n+1)+n)$*-sequential.*

Proof. Suppose that $V(H) = \{w_i \mid i = 1, 2, \ldots, 2n+1\}$ and h is an r-sequential labeling of H such that $h(w_i) = i-1$, $i = 1, 2, \ldots, 2n+1$. Let $u_{i,j}$ be the isomorphic image of u_j which is in G_i attached to H at w_i, $i = 1, 2, \ldots, 2n+1$.

In case (a), we let $u_{i,0} = w_i$, $i = 1, 2, \ldots, 2n+1$, where $g(u_0) = (k-1)/2$. Now, we are ready to give a sequential labeling f of $H \oplus [G]_{u_0}$.

Let f be defined as follows:

$$f(u_{i,j}) = \begin{cases} g(u_j)(2n+1)+i-1, & \text{if } d(u_0,u_j) \text{ is even and } 1 \le i \le 2n+1; \\ g(u_j)(2n+1)+n-(i-1)/2, & \text{if } d(u_0,u_j) \text{ is odd and } i = 1,3,\ldots,2n+1; \text{ and} \\ g(u_j)(2n+1)+2n+1-i/2, & \text{if } d(u_0,u_j) \text{ is odd and } i = 2,4,\ldots,2n. \end{cases}$$

One can easily observe that all the vertex labels are consecutive integers from 0 to $p(2n+1)-1$. A routine computation shows that if $u_{j_1}u_{j_2} \in E(G)$, then the set of the edge labels of its isomorphic images $u_{i,j_1}u_{i,j_2}$, $i = 1, 2, \ldots, 2n+1$, is $\{ f^{\#}(u_{i,j_1}u_{i,j_2}) | i = 1, 2, \ldots, 2n+1 \} = \{ g^{\#}(u_{j_1}u_{j_2})(2n+1)+l | l = n, n+1, \ldots, 3n \}$. So, the edge labels of all G_i's induced by f are consecutive integers from $k(2n+1)+n$ to $(k+p-2)(2n+1)+3n$. Next, let us consider the edge labels of H. Since $f(w_i) = f(u_{i,0}) = (k-1)(2n+1)/2 + i - 1 = (k-1)(2n+1)/2 + h(w_i)$, we have $f^{\#}(e) = (k-1)(2n+1) + h^{\#}(e)$ for each edge $e \in E(H)$. Hence, the edge labels of H induced by f are $k(2n+1)-n$, $k(2n+1)-n+1$, $\ldots$, $k(2n+1)+n-1$. Therefore, f is a $(k(2n+1)-n)$ - sequential labeling of $H \oplus [G]_{u_0}$.

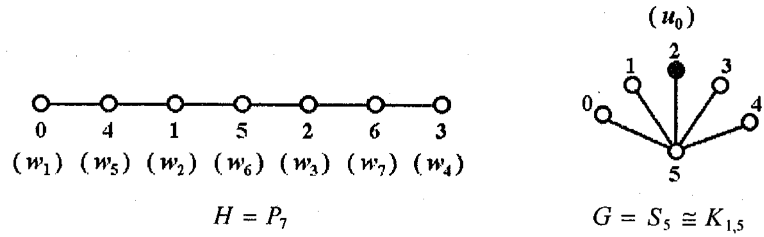
In case (b), if k is even and p is odd, then g^c is a $(p-k)$ - sequential labeling of G where $(p-k)$ is odd. In addition, $g^c(u_0) = (p-1)-(p+k-1)/2 = ((p-k)-1)/2$ By replacing g with g^c in (a), we obtain a $((p-k)(2n+1)-n)$ - sequential labeling of $H \oplus [G]_{u_0}$.

In case (c), $k+p$ is odd. Identifying $w_1, w_2, \ldots, w_{2n+1}$ with $u_{1,0}$, $u_{2,0}$, $\ldots$, $u_{2n+1,0}$ respectively, where $g(u_0) = (k+p-1)/2$, the labeling f in (a) is a $(k(2n+1)+n)$ - sequential labeling of $H \oplus [G]_{u_0}$.

In case (d), if kp is odd, then g^c is a $(p-k)$ - sequential labeling of G where $(p-k)$ is even. Since $g^c(u_0) = (p-1)-(k-1)/2 = ((p-k)+p-1)/2$, we have a $((p-k)(2n+1)+n)$ - sequential labeling of $H \oplus [G]_{u_0}$ by replacing g with g^c in (c). ■

For clarity, we give an example to explain how the labeling works. The following graph in Figure 2 is also known as a palm tree. Let $H = P_7$ and

$G = S_5 \cong K_{1,5}$. Then, it is easy to check that H is 4-sequential and G is 5-sequential. Therefore, the labeling can be obtained easily.



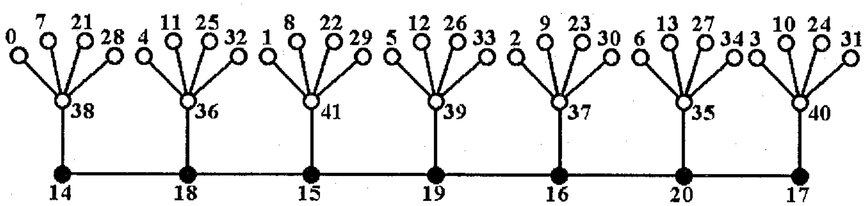


Figure 2. $P_7 \oplus [S_5]_{u_0}$ is 32-sequential.

If H and G are members of some special classes of graphs, then we can apply the "Attaching construction" directly. In what follows we present a case when H and G are caterpillars.

A caterpillar is a tree with the property that the removal of its pendant vertices leaves a path. It is well known that a caterpillar T is a bipartite graph (A, B). A $(k, p-k)$-caterpillar is a caterpillar $T = (A, B)$ where $|A| = k$ and $|B| = p-k$.

Now, we are ready to construct a new class of sequential graphs.

Theorem 2.2. *Suppose that H is an* (n,n) *or* $(n+1,n)$ *- caterpillar and G is a* $(k, p-k)$*-caterpillar, then* $H \oplus [G]$ *is* (np) *or* $(np+k)$*- sequential respectively.*

Proof. Let G be a $(k, p-k)$-caterpillar with two partite sets $A = \{u_0, u_1, \ldots, u_{k-1}\}$ and $B = \{u_k, u_{k+1}, \ldots, u_{p-1}\}$.

Case (1) : H is an (n,n)-caterpillar. Suppose that $A' = \{w_1, w_2, \ldots, w_n\}$ and $B' = \{w_{n+1}, w_{n+2}, \ldots, w_{2n}\}$ are the two partite sets of H. Choose any vertex, say u_x, of G as the attaching vertex. Then w_i is identified with $u_{i,x}$ for i = 1, 2, …, $2n$. Let f be a labeling of $H \oplus [G]_{u_x}$ defined by

$$f(u_{i,j}) = \begin{cases} j+(i-1)p, & \text{if } i = 1,2,\dots,n, \quad \text{and } j = 0,1,\dots,k-1; \\ (n+i-1)p+j-k, & \text{if } i = 1,2,\dots,n, \quad \text{and } j = k,k+1,\dots,p-1; \\ ip-j-1, & \text{if } i = n+1,n+2,\dots,2n, \quad \text{and } j = 0,1,\dots,k-1; \text{ and} \\ (i-n)p+k-j-1, & \text{if } i = n+1,n+2,\dots,2n, \quad \text{and } j = k,k+1,\dots,p-1. \end{cases}$$

It's not difficult to check that $f^{\#}(E(G_i)) =$

$$\begin{cases} \{(n+2(i-1))p,\ (n+2(i-1))p+1,\dots,\ (n+1+2(i-1))p-2\}, & \text{if } i = 1,2,\dots,n, \\ \{(2i-n-1)p,\ (2i-n-1)p+1,\dots,\ (2i-n)p-2\}, & \text{if } i = n+1,n+2,\dots,2n. \end{cases}$$

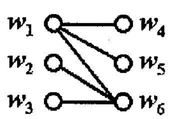
In addition, one can verify that the edge labels of H are $(n+1)p-1$, $(n+2)p-1$, …, $(3n-1)p-1$. So the edge labels of $H \oplus [G]_{u_x}$ are consecutive integers from np to $3np-2$.

Case (2): H is an $(n+1,n)$-caterpillar. Let $A' = \{w_1, w_2,\dots, w_{n+1}\}$ and $B' = \{w_{n+2}, w_{n+3},\dots, w_{2n+1}\}$ be the two partite sets of H. We identify w_i with $u_{i,x}$ for i = 1, 2, …, $2n+1$, where u_x is any vertex of G. The desired labeling f of $H \oplus [G]_{u_x}$ is given by

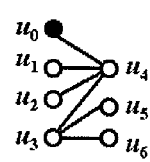
$$f(u_{i,j}) = \begin{cases} j+(i-1)p, & \text{if } i = 1,2,\dots,n+1, \quad j = 0,1,\dots,k-1; \\ (n+i-1)p+j, & \text{if } i = 1,2,\dots,n+1, \quad j = k,k+1,\dots,p-1; \\ (i-1)p+k-j-1, & \text{if } i = n+2,n+3,\dots,2n+1, \quad j = 0,1,\dots,k-1; \text{ and} \\ (i-n-1)p+k-j-1, & \text{if } i = n+2,n+3,\dots,2n+1, \quad j = k,k+1,\dots,p-1. \end{cases}$$

By a similar argument, we can show that f is an $(np+k)$- sequential labeling of $H \oplus [G]_{u_x}$. The proof is completed. ■

For clarity, we present an example in Figure 3 to depict the above construction.



H is a (3,3)-caterpillar.



G is a (4,3)-caterpillar.

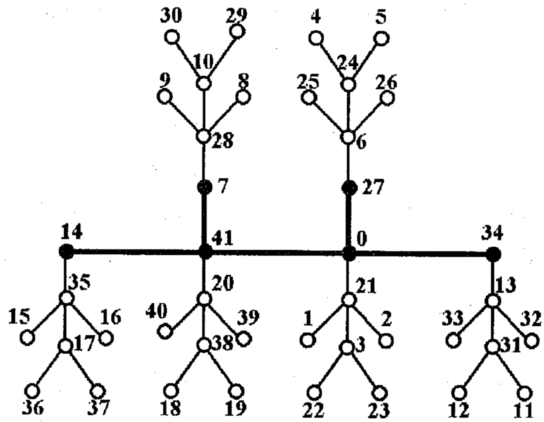


Figure 3. $H \oplus [G]_{u_0}$ is 21- sequential.

Remark: By making use of Theorem 2.2, we can also have a sequential labeling of a palm tree.

3. Adjoining Construction

The adjoining construction is in fact a special attaching-type operation. Instead of attaching a collection of vertex-disjoint graphs at each vertex of H respectively, the adjoining operation attaches all the vertex-disjoint graphs at each pendant vertex of a star. In other words, we adjoin a vertex w to a vertex v_i of each graph G_i in the collection of graphs $\{G_i\}$. We shall use $\oplus(G_1, G_2, \ldots, G_n)$ at $(v_1, v_2, \ldots, v_n)$ to denote the graph, see Figure 4. In the case that each $G_i \cong G$, i= 1, 2,..., n, and $v_1, v_2, \ldots, v_n$ are isomorphic images of the vertex v in $V(G)$, the graph is denoted by $\oplus G^n\big|_v$. Furthermore, we use $\oplus G^n$ to denote the collection of $|V(G)|$ graphs $\oplus G^n\big|_v$ where $v \in V(G)$.

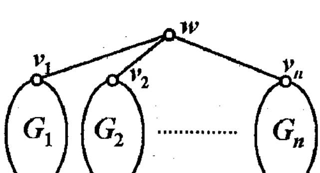


Figure 4. $\oplus(G_1, G_2, \ldots, G_n)$ at $(v_1, v_2, \ldots, v_n)$

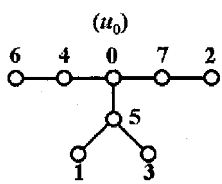
We recall that there are two types of sequential labeling for trees and the one we use above is of type 2. We shall use sequential labeling of type 1 in what follows.

Theorem 3.1. *If G is a k-sequential tree, then $\oplus G^{2n+1}$ is a $(k(2n+1)-n-1)$ or $(k(2n+1)+n)$ - sequential tree.*

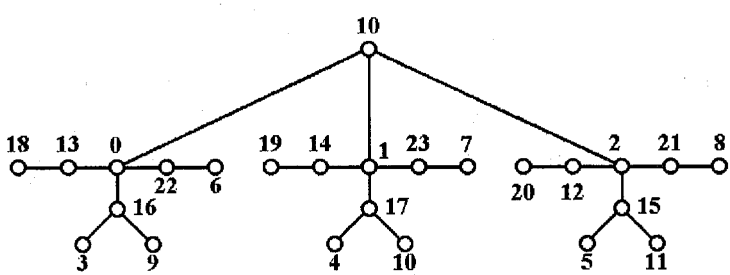
Proof. Let G be a k-sequential tree defined on $\{u_x \mid x \in \mathbf{Z}_p\}$ and the graph $\oplus G^{2n+1}$ is defined on $\{u_{i,x} \mid x \in \mathbf{Z}_p$ and $i = 1, 2, \ldots, 2n+1\} \cup \{w\}$ such that for each i, $u_{i,x}$ is an isomorphic image of u_x, $i = 1, 2, \ldots, 2n+1$, and w is adjoined to u_{i,x_0}, $i = 1, 2, \ldots, 2n+1$.

Now, by labeling each G as the labeling of G in Theorem 2.1 and labeling w by $(k - g(u_{x_0}))(2n+1) - n - 1$ or $(k + p - 1 - g(u_{x_0}))(2n+1) + n$ depending on the cases $0 \le g(u_{x_0}) \le k-1$ or $k \le g(u_{x_0}) \le p-1$, we have the desired labeling. ∎

For clarity, we present an example in Figure 5.



(1) G is 4-sequential.



(2) $\oplus G^3\big|_{u_0}$ is 10- sequential.

Figure 5.

Since P_m is sequential, we have the following result by Theorem 3.1.

Corollary 3.2. $\oplus P_m^{2n+1}$ *is* $([m/2](2n+1)-n-1)$ *or* $([m/2](2n+1)+n)$ *- sequential.*

Remark: In [1], Cahit uses the notation $S(t,m)$ to denote a star tree consisting of m branches where each branch is a path of length t. When we choose an endpoint v as the attaching vertex, $\oplus P_m^{2n+1}\big|_v$ is the same as $S(m,2n+1)$. Hence, Theorem 3.1 generalizes the result in [1].

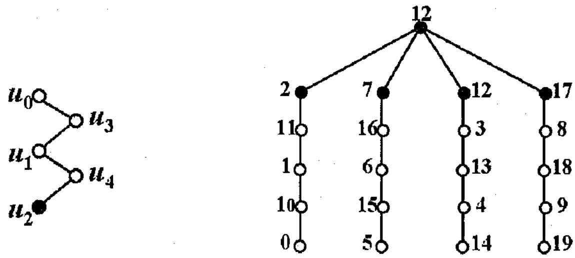
We next discuss the case when an even number of trees are adjoined to a vertex. We shall restrict the construction to adjoining $(k,p-k)$-caterpillars with partite sets $A=\{u_0, u_1, \ldots, u_{k-1}\}$ and $B=\{u_k, u_{k+1},\ldots, u_{p-1}\}$. In the following theorem, $\oplus G^{2n}\big|_{(x,y)}$ denotes the graph $\oplus G^{2n}$ at $(v_1,v_2,\ldots,v_{2n})$ in which $v_i=u_{i,x}$ for $i=1, 2,\ldots, n$, and $v_i=u_{i,y}$ for $i=n+1$, $n+2$, ..., $2n$, where $0\le x,y\le p-1$.

Theorem 3.3. *Let G be a* $(k,p-k)$*- caterpillar with* $k>p/2$. *If* $p-k\le x\le k-1$ *and* $y=p-x-1$, *then* $\oplus G^{2n}\big|_{(x,y)}$ *is an* (np)*- sequential tree.*

Proof. We label the $2n$ copies of G by using the same labeling as in case (1) of Theorem 2.2 and label the new vertex w by $(n+1)p-1-x$. Then the edge labels of all G_i's, as shown in Theorem 2.2, are $(n-1+l)p$, $(n-1+l)p+1, \ldots,(n+l)p-2$, $l=1, 2, \ldots, 2n$. In addition, the labels of the new edges wv_i's are $(n+1)p-1$, $(n+2)p-1$, ..., $3np-1$. Therefore, the graph $\oplus G^{2n}\big|_{(x,y)}$ is (np)-sequential. ∎

Corollary 3.4. *If G is a* $(k,2m+1-k)$*- caterpillar with* $k\ge m+1$, *then* $\oplus G^{2n}\big|_{u_m}$ *is a* $(2m+1)n$*-sequential tree.*

Remark: Since P_{2m+1} is an $(m+1,m)$-caterpillar, $\oplus P_{2m+1}^{2n}$ is a $(2m+1)n$-sequential tree in which the root (the new vertex) has the repeated label. Figure 6(2) shows a labeling of $\oplus P_5^4\big|_{u_2}$. As mentioned above, $\oplus P_{2m+1}^{2n}\big|_{u_m}$ is in fact the graph $S(2m+1,2n)$ defined in [1]. Thus, Theorem 3.3 also disproves the claim given at the end of [1]: when $p\equiv 0 \pmod 2$ and $k\ge 3$, there exists no harmonious labeling of $S(k,p)$ in which the root has the repeated label.



(1) P_5 (2) $\oplus P_5^4\big|_{u_2}$ is 10-sequential.

Figure 6.

Finally, we introduce a family of trees called "*fireworks*" in which graphs are adjoined nicely.

Adopting the notation $S(t,m)$ used in [1], the family $\oplus(S(t,m))^n\big|_v$, where v is the center of $S(t,m)$, of trees are called "*fireworks*" and denoted by $F(n,m,t)$. The following figure shows the graph $F(2,3,4)$.

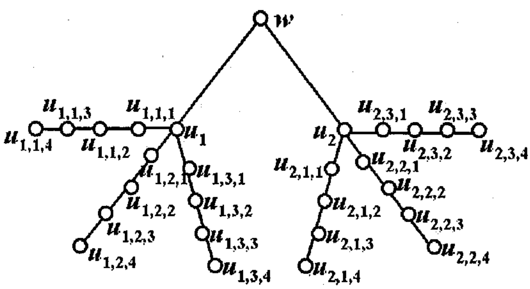


Figure 7. $\oplus(S(4,3))^2\big|_v = F(2,3,4)$

In a tree $F(n,m,t)$, the following notation for the vertices, as shown in Figure 7, is used throughout the rest of the paper: We call the center w, and in the ith copy of $S(t,m)$ we let u_i, $u_{i,j,1}$, ..., $u_{i,j,t}$ be consecutive vertices of the jth path where u_i is the common vertex of the m paths, $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, n$.

To prove the next theorem, the following result obtained in [6] is used.

Theorem 3.5.[6] *If G is a harmonious tree with no repeated vertex labels, then the one point union of an odd number of copies of G using the vertex labeled 0 as the common vertex is harmonious with no repeated vertex labels.*

Now, we have the results on "fireworks".

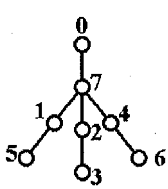
Theorem 3.6. $F(a,b,t)$ *is harmonious of type* 2 *if one of the following conditions satisfies:*
(1) *a is odd and b is even,* (2) $t=2$ *and ab is odd or a is an odd even number and* (3) $t=1$ *and a is odd or both a and b are even.*

Proof. Case (1): a is odd and b is even. Since $F(2n+1,2m,t)$ is the one point union of $2n+1$ copies of $F(1,2m,t)$, we start with a labeling $\bar{f}$ of $F(1,2m,t)$ as follows:

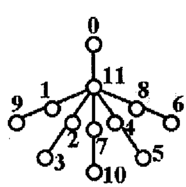
$$\bar{f}(u)=\begin{cases}0, & \text{if } u=w;\\ 2tm+1, & \text{if } u=u_{1,0};\\ (t(j-1)+x)/2, & \text{if } u=u_{1,j,x}, j=1,3,\dots,2m-1, \text{ and } x=2,4,\dots,2[t/2];\\ (tj-x+1)/2, & \text{if } u=u_{1,j,x}, j=2,4,\dots,2m, \text{ and } x=1,3,\dots,2\left[\frac{t-1}{2}\right]+1;\\ (t(2m+j-1)+x+1)/2, & \text{if } u=u_{1,j,x}, j=1,3,\dots,2m-1, \text{ and } x=1,3,\dots,2\left[\frac{t-1}{2}\right]+1;\\ (t(2m+j)-x)/2+1, & \text{if } u=u_{1,j,x}, j=2,4,\dots,2m, \text{ and } x=2,4,\dots,2[t/2].\end{cases}$$

It is a routine matter to check that $\bar{f}$ is a harmonious labeling of type 2 of $F(1,2m,t)$. So, the graph $F(2n+1,2m,t)$ obtained by identifying the vertices labeled 0 (which is the center w) of the $(2n+1)$ copies of $F(1,2m,t)$ is harmonious of type 2 by Theorem 3.5. See Figure 8(1) for an example. (Note that in Figure 8(1), every vertex label of the $(i+1)$th copy of $F(1,4,3)$ is in fact obtained by adding 13 to the label of its isomorphic vertex in the ith copy.)

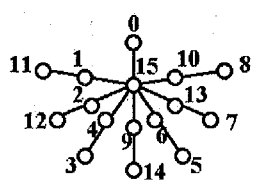
Case (2): (i) t = 2 and ab is odd. First, we construct a labeling of $F(1,2m+1,2)$. Due to different structures, the harmonious labelings of $F(1,3,2)$, $F(1,5,2)$ and $F(1,7,2)$ are given individually in the following.



$F(1,3,2)$



$F(1,5,2)$



$F(1,7,2)$

For $m \ge 4$, the labeling $\overline{f}$ of $F(1,2m+1,2)$ is defined by: $(x=1,2)$

$$\overline{f}(u)=\begin{cases} 0, & \text{if } u=w; \\ 4m+3, & \text{if } u=u_{1,0}; \\ j+2(m+2)(x-1), & \text{if } u=u_{1,j,x}, \text{ and } j=1,2,4,6,\ldots,2[(m-1)/2]; \\ j+2(m+2)(2-x), & \text{if } u=u_{1,j,x}, \text{ and } j=3,5,7,\ldots,2[m/2]-1; \\ j+\left|x-\left(3-(-1)^m\right)/2\right|, & \text{if } u=u_{1,j,x}, \text{ and } j=m,m+2; \\ 2m+3+(2m-1)(x-1), & \text{if } u=u_{1,m+1,x}; \\ j+1+2m(x-1), & \text{if } u=u_{1,j,x},\ j=2\left\lceil\frac{m-1}{2}\right\rceil+5, 2\left\lceil\frac{m-1}{2}\right\rceil+7,\ldots,2m-1; \\ j+1+2m(2-x), & \text{if } u=u_{1,j,x}, \text{ and } j=2[m/2]+4, 2[m/2]+6,\ldots,2m; \\ 2(m+1)+2(2-x), & \text{if } u=u_{1,2m+1,x}. \end{cases}$$

Then $F(1,2m+1,2)$, $m \ge 1$, is harmonious of type 2 and w is the vertex labeled 0. So, $F(2n+1,2m+1,2)$ is harmonious of type 2 by Theorem 3.5. (See Figure 8(2) for an example.)

(ii) $t=2$ and $a=4n+2$, we start with a labeling $\overline{f}$ of $F(2,m,2)$:

$$\overline{f}(u)=\begin{cases} 0, & \text{if } u=w; \\ (2m+1)i, & \text{if } u=u_{i,0},\ i=1,2; \\ j+(i-1)m, & \text{if } u=u_{i,j,1},\ i=1,2, \text{ and } 1 \le j \le m; \text{ and} \\ 2(2m-j+1)+(i-1), & \text{if } u=u_{i,j,2},\ i=1,2, \text{ and } 1 \le j \le m. \end{cases}$$

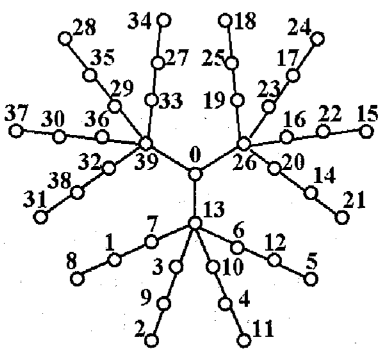
One can easily verify that $F(4n+2,m,2)$ is harmonious of type 2. (See Figure 8(3) for an example.)

Case (3): (i) $t=1$ and a is odd. $F(2n+1,m,1)$ is exactly the graph $\oplus (S_m)^{2n+1}\big|_v$, where v is the center of the star S_m. By Theorem 3.1, it is harmonious of type 1. On the other hand, we can also obtain a harmonious labeling of type 2. First, give $F(1,m,1)=S_{m+1}$ a harmonious labeling with 0 on a pendant vertex. By identifying the vertices labeled 0 in the $2n+1$ copies of $F(1,m,1)$, we obtain the harmonious graph $F(2n+1,m,1)$ of type 2 by Theorem 3.5, see Figure 8(4) for an example.

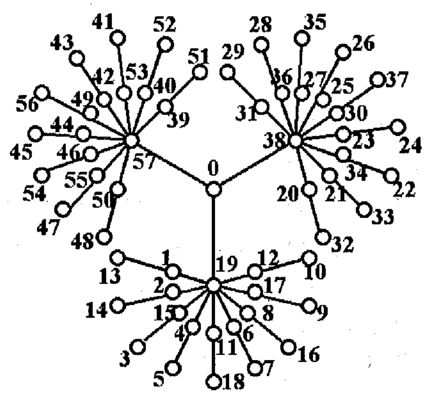
(ii) $t=1$ and both a and b are even. We define a labeling f on $F(2n,2m,1)$ as follows:

$$f(u)=\begin{cases} 0, & \text{if } u=w\,; \\ i(2m+1), & \text{if } u=u_{i,0},\quad 1\le i\le 2n\,; \\ 2n+1-i+2(j-1), & \text{if } u=u_{i,j,1},\quad i=2n-1,2n, \text{ and } 1\le j\le m\,; \\ (2m+3)n+1-i+2(j-m-1), & \text{if } u=u_{i,j,1},\quad i=2n-1,2n, \text{ and } m+1\le j\le 2m\,; \\ f(u_{i+2,j,1})+(2m+1), & \text{if } u=u_{i,j,1},\quad 1\le i\le 2n-2, \text{ and } 1\le j\le 2. \end{cases}$$

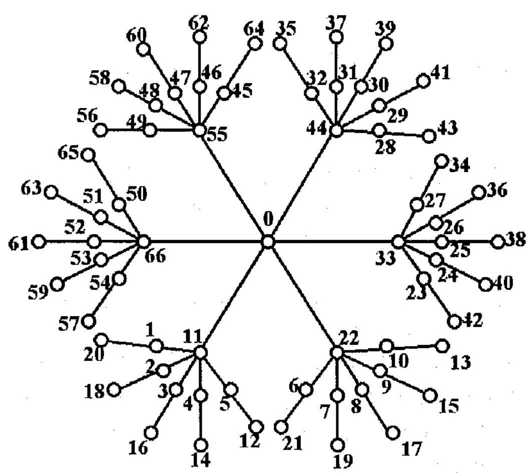
A routine verification shows that f is a harmonious labeling of type 2 of $F(2n,2m,1)$. (See Figure 8(5) for an example.) ∎



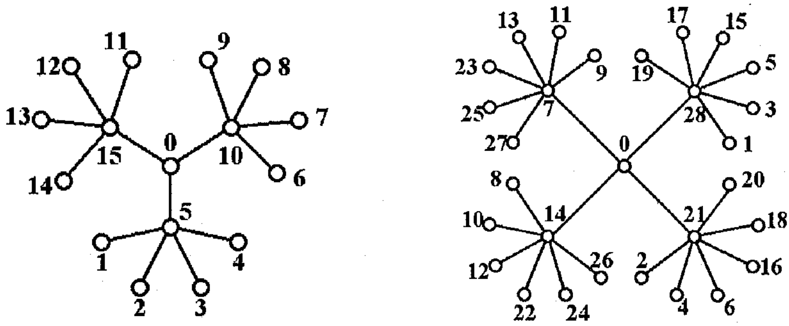
(1) $F(3,4,3)$ (obtained from $F(1,4,3)$)



(2) $F(3,9,2)$ (obtained from $F(1,9,2)$)



(3) $F(6,5,2)$ (obtained from $F(2,5,2)$)



(4) $F(3,4,1)$ (obtained from $F(1,4,1)$) (5) $F(4,6,1)$

Figure 8.

Conclusion

We manage to construct a couple of classes of harmonious trees and the graphs in the first class are in fact sequential graphs which can be utilized to construct new harmonious graphs recursively. But, the second class does not have the extra property we would like to have. If we can obtain a sequential labeling of type 2 for the graphs in the second class, then better results can be obtained, for example showing the complete q-ary trees are harmonious. So far, using the result on firework, we are able to show that for every q, every complete q-ary tree with two levels $F(q,q,1)$ is harmonious (of type 2).

Acknowledgment

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