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A Note on Cyclic m-cycle Systems of $K_{r(m)}$

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Abstract. It was proved by Buratti and Del Fra that for each pair of odd integers r and m, there exists a cyclic m-cycle system of the balanced complete r-partite graph $K_{r(m)}$ except for the case when $r = m = 3$. In this note, we study the existence of a cyclic m-cycle system of $K_{r(m)}$ where r or m is even. Combining the work of Buratti and Del Fra, we prove that cyclic m-cycle systems of $K_{r(m)}$ exist if and only if (a) $K_{r(m)}$ is an even graph (b) $(r, m) \neq (3, 3)$ and (c) $(r, m) \not\equiv (t\ , 2) \pmod 4$ where $t \in \{2, 3\}$.

The Main Result

An *m-cycle system* of a simple graph G is a set $\boldsymbol{C}$ of edge disjoint m-cycles which partition the edge set of G. The necessary conditions for the existence of an m-cycle system of a graph G are that the value of m is not exceeding the order of G, m divides the number of edges in G, and the degree of each vertex in G is even. An *even* graph is a graph with vertex degrees all even. If G is a complete graph K_v on v vertices, then it is called an m-cycle system of order v.

Alspach and Gavlas [1] and Šajna [11] have completely settled the existence problem of m-cycle systems of K_v and $K_v - I$, where I is a 1-factor. Moreover, there have been many results in the literature concerning the existence problem of cyclic m-cycle systems. The reader can refer to [2–10, 12, 13].

A graph G is said to be a *complete r-partite graph* ($r > 1$) if its vertex set V can be partitioned into r disjoint non-empty sets $V_1, \ldots, V_r$ (called *partite sets*) such that there exists exactly one edge between each pair of vertices from different partite sets. If $|V_i| = n_i$ for $1 \le i \le r$, the complete r-partite graph is denoted by $K_{n_1,\cdots,n_r}$. In particular, if $n_1 = \cdots = n_r = k(> 1)$, it is called *balanced* and the graph will be simply denoted by $K_{r(k)}$.

Let $C = (c_0, \ldots, c_{m-1})$ be an m-cycle. An m-cycle system of a graph G, $\boldsymbol{C}$, is said to be *cyclic* if $V(G) = Z_v$ and $(c_0 + 1, \ldots, c_{m-1} + 1) \in \boldsymbol{C} \pmod v$ whenever $(c_0, \ldots, c_{m-1}) \in \boldsymbol{C}$.

The necessary conditions for the existence of a cyclic m-cycle system of a complete r-partite graph G are that G is balanced, say $K_{r(k)}$, and any partite set in $K_{r(k)}$ is the subgroup

$$rZ_k = \{0, r, \ldots, (k-1)r\}$$

of Z_{rk} or its coset. For $i = 0, \ldots, r-1$, let V_i denote the ith partite set of $K_{r(k)}$. We may assume $V_i = \{i, i+r, i+2r, \ldots, i+(k-1)r\}$. Note that the set of distinct differences of edges in $K_{r(k)}$ is $Z_{rk} \setminus \pm \{0, r, 2r, \ldots, \lfloor k/2 \rfloor r\}$.

For any edge $\{a, b\}$ in G with $V(G) = Z_v$, we shall use $\pm |a-b|$ to denote the *difference* of the edge $\{a, b\}$. The number of distinct differences in a cycle C is called the *weight* of C.

Let $m = ab$ be a positive integer (> 2). An m-cycle C in $K_{r(k)}$ with weight a has *index* $\frac{rk}{b}$ if for each edge $\{s, t\}$ in C, the edges $\{s + i \cdot \frac{rk}{b}, t + i \cdot \frac{rk}{b}\}$ (mod rk) with $i \in Z_b$ are also in C.

Proposition 1 ([14]). *Let $m = ab$ be a positive integer (> 2). Then there exists an m-cycle $C = (c_0, \ldots, c_{m-1})$ in $K_{r(k)}$ with weight a and index $\frac{rk}{b}$ if and only if each of the following conditions is satisfied:*

(1) for $0 \le i \ne j \le a-1$, $c_i \not\equiv c_j \pmod{\frac{rk}{b}}$;
(2) the differences of the edges $\{c_i, c_{i-1}\}$ $(1 \le i \le a)$ are all distinct;
(3) $c_a = c_0 + t \cdot \frac{rk}{b}$, where $\gcd(t, b) = 1$; and
(4) $c_{ia+j} = c_j + i \cdot t \cdot \frac{rk}{b}$ where $0 \le j \le a-1$ and $0 \le i \le b-1$.

To simplify, the m-cycle $C = (c_0, \ldots, c_{a-1}, c_0 + t \cdot \frac{rk}{b}, \ldots, c_{a-1} + t \cdot \frac{rk}{b}, \ldots, c_0 + (b-1) \cdot t \cdot \frac{rk}{b}, \ldots, c_{a-1} + (b-1) \cdot t \cdot \frac{rk}{b})$ will be denoted by $C = [c_0, \ldots, c_{a-1}]_{t \cdot rk/b}$.

Note that if C is any cycle with weight a in a cyclic m-cycle system of $K_{r(k)}$, then C is precisely an m-cycle with index $\frac{rk}{b}$.

The following results are either known or easy to verify, we list them without the details of proof.

Theorem 2 ([2]). *For each pair of odd integers r and m, there exists a cyclic m-cycle system of $K_{r(m)}$ with the exception that $(r, m) = (3, 3)$.*

Lemma 3 ([14]). *If there is a cyclic m-cycle system of a graph G, then G is $2r$-regular for some positive integer r.*

Proposition 4 ([14]). *If there is a cyclic m-cycle system of $K_{r(m)}$ with m even and $m > 4$, then $(r, m) \not\equiv (t, 2) \pmod 4$ where $t \in \{2, 3\}$.*

Note that if m is odd, then r must be odd since $K_{r(m)}$ is an even graph.

For an m-cycle $C = (c_0, \ldots, c_{m-1})$, we shall use ∂C to denote the set of distinct differences $\{\pm(c_i - c_{i-1}) | i = 1, \ldots, m\}$ where $c_m = c_0$. Given a set $D = \{C_1, \ldots, C_p\}$ of m-cycles, the list of differences from D is defined as the union of the multisets $\partial C_1, \ldots, \partial C_p$, i.e., $\partial D = \bigcup_{i=1}^{p} \partial C_i$.

Theorem 5 ([14]). *Let D be a set of m-cycles with vertices in Z_{rk} such that $\partial D = Z_{rk} \setminus \pm \{0, r, 2r, \ldots, \lfloor k/2 \rfloor r\}$. Then there exists a cyclic m-cycle system of $K_{r(k)}$.*

We are now ready for the main result. First, we will assume $C_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,2s}, v_{i,2s+1}, v_{i,2s'}, v_{i,2s-1'}, \ldots, v_{i,1'})$ to be a $(4s+2)$-cycle, and an m-cycle with weight m is called *full*, otherwise *short*.

Theorem 6. *A cyclic m-cycle system of* $K_{r(m)}$ *exists if and only if (a)* $K_{r(m)}$ *is an even graph (b)* $(r, m) \neq (3, 3)$ *and (c)* $(r, m) \not\equiv (t, 2) \pmod 4$ *where* $t \in \{2, 3\}$.

Proof. The necessary part follows by Theorem 2 and Proposition 4. Therefore, we prove the sufficiency in what follows. The proof is split into 4 cases: (i) $(r, m) \equiv (0, 2) \pmod 4$ (ii) $(r, m) \equiv (1, 2) \pmod 4$ (iii) $r \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 4$ (iv) $r \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 4$. Note that if m is odd, then r must be odd and this case has been settled by Buratti and Del Fra in [2].

Case 1. $(r, m) \equiv (0, 2) \pmod 4$.

Subcase 1.1. $r \equiv 0 \pmod 4$ and $m \equiv 2 \pmod 8$, say $r = 4p$ and $m = 8k + 2$.

Let $C^* = [c_0, \ldots, c_{4k}]_{r(4k+1)}$ be a short m-cycle defined as

$$c_i = \begin{cases} 2rj, \text{ if } i = 2j \text{ with } j = 0, \ldots, 2k-1; \\ 4r - 1 + 8jr, \text{ if } i = 4j+1 \text{ with } j = 0, \ldots, k-1; \\ 7r - 1 + 8jr, \text{ if } i = 4j\ +3 \text{ with } j = 0, \ldots, k-2; \\ r(8k-1)+1, \text{ if } i = 4k-1; \text{ and} \\ 4rk + 2, \text{ if } i = 4k. \end{cases}$$

It can be checked that all values in C^* are certainly pairwise distinct and the set of differences occurring in C^* is $\partial C^* = \pm\{r-2, 2r-1, 3r-1, \ldots, (4k+1)r-1\}$.

For $i = 1, \ldots, p$, the full m-cycles C_i are defined as

$v_{i,0} = 0$; for $j = 0, \ldots, 2k-1$, $v_{i,2j+1} = jr - 3 + 4i$, $v_{i,2j+1'} = v_{i,2j+1} + 2$; for $j = 1, \ldots, 2k-1$, $v_{i,2j} = r(4k+1-j) - 6 + 8i$, $v_{i,2j'} = v_{i,2j} + 3$; $v_{i,4k} = r(2k+1) - 5 + 8i$, $v_{i,4k'} = v_{i,4k} + 14$; and $v_{i,4k+1} = 2rk - 2 + 4i$.

If $p \geq 2$, then for $i = 1, \ldots, p-1$, the remaining full m -cycles C_{p+i} are given by

$v_{p+i,0} = 0$; for $j = 0, \ldots, 2k-1$, $v_{p+i,2j+1} = jr - 2 + 4i$, $v_{p+i,2j+1'} = v_{p+i,2j+1} + 2$; for $j = 1, \ldots, 2k-1$, $v_{p+i,2j} = r(4k+1-j) - 3 + 8i$, $v_{p+i,2j'} = v_{p+i,2j} + 3$; $v_{p+i,4k} = r(2k+1) - 2 + 8i$, $v_{p+i,4k'} = v_{p+i,4k} + 1$; and $v_{p+i,4k+1} = 2rk - 1 + 4i$.

By routine computation, we have that all values in each full m-cycle constructed above are also pairwise distinct and $\bigcup_{i=1}^{2p-1} \partial C_i = \pm\{1, 2, \ldots, r-3, r-1\} \cup \bigcup_{i=0}^{4k-1} \pm\{r+1+ir, r+2+ir, \ldots, 2r-2+ir\}$.

Since $\partial C^* \cup \bigcup_{i=1}^{2p-1} \partial C_i = Z_{rm} \setminus \pm\{0, r, 2r, \ldots, rm/2\}$, it follows from Theorem 5 that there exists a cyclic m-cycle system of $K_{r(m)}$.

Subcase 1.2. $r \equiv 0 \pmod 4$ and $m \equiv 6 \pmod 8$, say $r = 4p$ and $m = 8k + 6$.

If $k = 0$, then $C^* = [0, 4r-1, 3r-2]_{3r}$ is the short 6-cycle. For $i = 1, \dots, p$, the full 6-cycles are $C_i = (0, 4i-3, 2r-4+8i, r-2+4i, 2r-3+8i, 4i-1)$ and, if $p \ge 2$, for $i = 1, \dots, p-1$, the remaining full 6-cycles are $C_{p+i} = (0, 4i, 2r+1+8i, r+1+4i, 2r+2+8i, 4i+2)$.

We then have that $\partial C^* = \pm\{2, r+1, 2r+1\}$ and $\bigcup_{i=1}^{2p-1} \partial C_i = \pm\{1, 3, 4, \dots, r-1, r+2, r+3, \dots, 2r-1, 2r+2, 2r+3, \dots, 3r-1\}$.

If $k > 0$, then the short m-cycle $C^* = [c_0, \dots, c_{4k+2}]_{r(4k+3)}$ is defined as

$$c_i = \begin{cases} 2jr, \text{ if } i = 2j \text{ with } j = 0, \dots, 2k; \\ 3r+1+8jr, \text{ if } i = 4j+1 \text{ with } j = 0, \dots, k-1; \\ 6r+1+8jr, \text{ if } i = 4j+3 \text{ with } j = 0, \dots, k-1; \\ 4r(2k+1)-1, \text{ if } i = 4k+1; \text{ and} \\ r(4k+3)-2, \text{ if } i = 4k+2, \end{cases}$$

and $\partial C^* = \pm\{2, r+1, 2r+1, \dots, (4k+2)r+1\}$.

For $i = 1, \dots, p$, the full m-cycles C_i are defined as

$v_{i,0} = 0$; for $j = 0, \dots, 2k$, $v_{i,2j+1} = jr-3+4i$, $v_{i,2j+1'} = v_{i,2j+1}+2$, $v_{i,2j+2} = r(4k+2-j)-4+8i$, and $v_{i,2j+2'} = v_{i,2j+2}+1$; and $v_{i,4k+3} = r(2k+1)-2+4i$.

For $i = 1, \dots, p-1$, the rest of full m-cycles C_{p+i} are given by

$v_{p+i,0} = 0$; for $j = 0, \dots, 2k$, $v_{p+i,2j+1} = jr+4i$, $v_{p+i,2j+1'} = v_{p+i,2j+1}+2$, $v_{p+i,2j+2} = r(4k+2-j)+1+8i$, and $v_{p+i,2j+2'} = v_{p+i,2j+2}+1$; and $v_{p+i,4k+3} = r(2k+1)+1+4i$.

An easy verification shows that $\bigcup_{i=1}^{2p-1} \partial C_i = \pm\{1, 3, 4, \dots, r-1\} \cup \bigcup_{i=0}^{4k+1} \pm\{(r+2+ir, r+3+ir, \dots, 2r-1+ir\}$.

Case 2. $r \equiv 1 \pmod 4$ and $m \equiv 2 \pmod 4$, say $r = 4p+1$ and $m = 4k+2$.

It suffices to consider the full m- cycles.

For $i = 1, \dots, p$, the full m-cycles C_i are defined as

$v_{i,0} = 0$; for $j = 0, \dots, k-1$, $v_{i,2j+1} = jr-3+4i$, $v_{i,2j+1'} = v_{i,2j+1}+2$; for $j = 1, \dots, k-1$ (if $k \ge 2$), $v_{i,2j} = r(2k+1-j)-6+8i$, $v_{i,2j'} = v_{i,2j}+3$; $v_{i,2k} = r(k+1)-5+8i$, $v_{i,2k'} = v_{i,2k}+1$; and $v_{i,2k+1} = rk-2+4i$.

We have $\bigcup_{i=1}^{p} \partial C_i = \pm\{1, 3, \dots, r-2\} \cup \bigcup_{i=0}^{p-1} \bigcup_{j=1}^{2k} \pm\{jr+1+4i, jr+2+4i\}$.

For $i = 1, \dots, p$, let C_{p+i} be the rest of the full m-cycles given by

$v_{p+i,0} = 0$; for $j = 0, \dots, k-1$, $v_{p+i,2j+1} = jr-2+4i$, $v_{p+i,2j+1'} = v_{p+i,2j+1}+2$; for $j = 1, \dots, k-1$, $v_{p+i,2j} = r(2k+1-j)-3+8i$, $v_{p+i,2j'} = v_{p+i,2j}+3$; $v_{p+i,2k} = r(k+1)-2+8i$, $v_{p+i,2k'} = v_{p+i,2k}+1$; and $v_{p+i,2k+1} = rk-1+4i$.

It follows that $\bigcup_{i=1}^{p} \partial C_{p+i} = \pm\{2, 4, \dots, r-1\} \cup \bigcup_{i=0}^{p-1} \bigcup_{j=1}^{2k} \pm\{jr+3+4i, jr+4+4i\}$, and $\bigcup_{i=1}^{2p} \partial C_i = Z_{rm} \setminus \pm\{0, r, 2r, \dots, (2k+1)r\}$.

Case 3. $r \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 4$.

Subcase 3.1. $m \equiv 0 \pmod 8$, say $m = 8k$.

For $i = 1, \ldots, r-1$, the short m-cycles $C_i^* = [c_{i,0}, \ldots, c_{i,4k-1}]_{4rk}$ are defined as

$$c_{i,j} = \begin{cases} 2rs, \text{ if } j = 2s \text{ with } s = 0, \ldots, 2k-1; \\ 2r + i + 8rs, \text{ if } j = 4s+1 \text{ with } s = 0, \ldots, k-1; \text{ and} \\ 5r + i + 8rs, \text{ if } j = 4s+3 \text{ with } s = 0, \ldots, k-1. \end{cases}$$

We have $\partial C_i^* = \pm\{i, r+i, 2r+i, \ldots, (4k-1)r+i\}$ and $\bigcup_{i=1}^{r-1} \partial C_i^* = Z_{rm} \setminus \pm \{0, r, 2r, \ldots, 4rk\}$.

Subcase 3.2. $m \equiv 4 \pmod 8$, say $m = 8k+4$.

For $i = 1, \ldots, r-1$, the short m-cycles $C_i^* = [c_{i,0}, \ldots, c_{i,4k+1}]_{r(4k+2)}$ are given by

$$c_{i,j} = \begin{cases} 2rs, \text{ if } j = 2s \text{ with } s = 0, \ldots, 2k; \\ r + i + 8rs, \text{ if } j = 4s+1 \text{ with } s = 0, \ldots, k; \text{ and} \\ 6r + i + 8rs, \text{ if } k \geq 1 \text{ and } j = 4s+3 \text{ with } s = 0, \ldots, k-1. \end{cases}$$

$\partial C_i^* = \pm\{r-i, r+i, 2r+i, 3r+i, \ldots, (4k+1)r+i\}$ and $\bigcup_{i=1}^{r-1} \partial C_i^* = Z_{rm} \setminus \pm \{0, r, 2r, \ldots, (4k+2)r\}$.

Case 4. $(r, m) \equiv (t, 0) \pmod 4$, $t \in \{1, 3\}$.

Subcase 4.1. $m \equiv 4 \pmod 8$, say $m = 8k+4$.

For $i = 1, \ldots, r-1$, the short m-cycles are $C_i^* = [0, i]_{2r}$ and $\bigcup_{i=1}^{r-1} \partial C_i^* = \pm\{1, 2, \ldots, r-1, r+1, r+2, \ldots, 2r-1\}$.

If $k \geq 1$ then for $i = 1, \ldots, r-1$, and $j = 1, \ldots, k$, the remaining short m-cycles are $C_{i,j}^* = [0, 4jr+i]_{2r}$ and $\hat{C}_{i,j}^* = [0, (4j+1)r+i]_{2r}$.

By routine computation, it follows that $\bigcup_{i=1}^{r-1} \bigcup_{j=1}^{k} \partial C_{i,j}^* = \bigcup_{s=0}^{k-1} \pm\{2r+1+4sr, 2r+2+4sr, \ldots, 3r-1+4sr, 4r+1+4sr, 4r+2+4sr, \ldots, 5r-1+4sr\}$ and $\bigcup_{i=1}^{r-1} \bigcup_{j=1}^{k} \partial \hat{C}_{i,j}^* = \bigcup_{s=0}^{k-1} \pm\{3r+1+4sr, 3r+2+4sr, \ldots, 4r-1+4sr, 5r+1+4sr, 5r+2+4sr, \ldots, 6r-1+4sr\}$.

Subcase 4.2. $m \equiv 0 \pmod 8$, say $m = 8k$.

For $i = 1, \ldots, r-1, andj = 1, \ldots, k$, the short m -cycles are $C_{i,j}^* = [0, (4j-2)r+i]_{2r}$ and $\hat{C}_{i,j}^* = [0, (4j-1)r+i]_{2r}$.

We have $\bigcup_{i=1}^{r-1} \bigcup_{j=1}^{k} \partial C_{i,j}^* = \bigcup_{s=0}^{k-1} \pm\{1+4sr, 2+4sr, \ldots, r-1+4sr, 2r+1+4sr, 2r+2+4sr, \ldots, 3r-1+4sr\}$ and $\bigcup_{i=1}^{r-1} \bigcup_{j=1}^{k} \partial \hat{C}_{i,j}^* = \bigcup_{s=0}^{k-1} \pm\{r+1+4sr, r+2+4sr, \ldots, 2r-1+4sr, 3r+1+4sr, 3r+2+4sr, \ldots, 4r-1+4sr\}$. □

We end this note with a conclusion. Assume m to be even (> 2) and $K_m - I$ to be the complete graph with 1-factor I removed. Observing the consequence of Theorem 6, it is clear that if there exists a cyclic m -cycle system of $K_m - I$, then a cyclic m-cycle system of $K_{rm} - I$ is given. Unfortunately, there does not exist a cyclic m-cycle system of $K_m - I$ except for $m = 4$.

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