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$\overrightarrow{C}_4$-Decompositions of $D_v \backslash P$ and $D_v \cup P$ where P is a 2-Regular Subgraph of D_v

Liqun Pu[1], Hung-Lin Fu[2] and Hao Shen[1,⋆]

[1] Department of Applied Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China. e-mail: liqunpu@yahoo.com.cn
[2] Department of Applied Mathematics, National Chiao Tung University, Hsin Chu 30050, Taiwan. e-mail: hlfu@math.nctu.edu.tw

Abstract. In this paper, we extend the study of C_4-decompositions of the complete graph with 2-regular leaves and paddings to directed versions. Mainly, we prove that if P is a vertex-disjoint union of directed cycles in a complete digraph D_v, then $D_v \setminus P$ and $D_v \cup P$ can be decomposed into directed 4-cycles, respectively, if and only if $v(v-1) - |E(P)| \equiv 0 \pmod 4$ and $v(v-1) + |E(P)| \equiv 0 \pmod 4$ where $|E(P)|$ denotes the number of directed edges of P, and $v \geq 8$.

Key words. Directed 4-cycles, Complete digraph, Packing, Covering

1. Introduction

A packing of a graph G with 4-cycles is a set of edge-disjoint 4-cycles in G. The graph induced by the edges in G but not in any 4-cycle of the packing is called the remainder graph of the packing or the leave of the packing. If a packing has a leave which has the minimum number of edges, we call it a minimum leave. A maximum packing of G (with 4-cycles) is a packing which has a minimum leave. Clearly, if $E(G)$ can be partitioned into sets which induce 4-cycles, then the leave is an empty graph and we say that G has a 4-cycle decomposition.

A 4-cycle decomposition of a complete graph K_v is also known as a 4-cycle system of order v. It is folklore that a 4-cycle system of order v exists if and only if $v \equiv 1 \pmod 8$ and the maximum packing of K_v [6] and $K_v \setminus P$ [2, 4] with 4-cycles where P is a special subgraph of K_v are also known. When H is a 2-regular subgraph of a complete graph K_{2m+1}, H. L. Fu and C. A. Rodger give us the following result.

Theorem 1 [4]. *Let H be a 2-regular subgraph of K_{2m+1} and $|E(H)|$ be the number of edges of H such that $\binom{2m+1}{2} - |E(H)|$ is a multiple of 4. Then $K_{2m+1} \setminus H$ has a 4-cycle decomposition.*

For convenience, we denote such a decomposition by $C_4 | K_{2m+1} \setminus H$.

⋆ *Present address:* Insert the address here if needed.

A covering of a graph G with 4-cycles is a collection of 4-cycles, $\mathcal{P}$, such that each edge of G occurs in at least one 4-cycle in $\mathcal{P}$. So, if $G(\mathcal{P})$ is the multigraph formed by joining each pair of vertices u and v with x edges if and only if $\mathcal{P}$ contains x 4-cycles that contain both u and v, then clearly $C_4 | G(\mathcal{P})$. The multigraph $G(\mathcal{P}) \setminus G$ is called the excess graph of G; it is also known as the padding of the covering $\mathcal{P}$ of G. A covering with smallest excess graph (in size) is called a minimum covering.

Similarly, packing and covering of complete digraph D_v with directed 4-cycles can also be defined. In this paper, we extend the work of Theorem 1 and consider the corresponding problem about packing and covering of a complete digraph with directed 4-cycles.

2. Preliminaries

Let D_v be a complete digraph without loops of order v and for each vertex w in D_v, $\deg^+(w) = \deg^-(w) = v - 1$. Let C_{l_i} be a cycle of length l_i, P be a vertex-disjoint union of cycles in D_v and $V(C_{l_i})$ be the set of vertices of C_{l_i}. Then $P_n = \cup_{i=1}^{k} C_{l_i}$, if $n = \sum_{i=1}^{k} l_i$, $V(C_{l_i}) \cap V(C_{l_j}) = \emptyset \, (i \neq j)$.

Let A be an m-set, B be an n-set and $A \cap B = \emptyset$. A complete bipartite directed graph $D_{A,B}$ $(D_{m,n})$ contains $2mn$ directed edges. For example, $D_{3,2} = \{a_i b_j, b_j a_i | 1 \leq i \leq 3,\ 1 \leq j \leq 2\}$, which has 12 edges. Note that $a_i b_j$ and $b_j a_i$ are two different directed edges.

Theorem 2 [5]. *For an integer v, $v \geq 8$, D_v has a $\vec{C}_4$-decomposition if and only if* $v \equiv 0, 1 \pmod 4$.

We denote such a decomposition by $\vec{C}_4 \, | D_v$.

The following lemma plays the most important role to prove our main theorem. Since the proof is easy to see, we omit the proof.

Lemma 1. *Let m, n be a positive integer such that min $\{m, n\} \geq 2$ and mn is even. Then $D_{m,n}$ can be decomposed into directed 4-cycles.*

In fact, this result is a special case of the following theorem, the well-known Sotteau's Theorem.

Theorem 3 [7]. *Let m, n be two positive integers such that min $\{m, n\} \geq k$ and $k|mn$. Then $D_{m,n}$ can be decomposed into directed $2k$-cycles.*

Lemma 2. *If a digraph G can be decomposed into directed 4-cycles and X is a k-set, $k \equiv 0 \pmod 4$ and $k \geq 8$, such that $V(G) \cap X = \emptyset$, then the graph $\widetilde{G} = G \vee D_k$ where $V(\widetilde{G}) = V(G) \cup X$ and $E(\widetilde{G}) = E(G) \cup E(X) \cup E(D_{V(G),X})$ has a directed 4-cycle decomposition.*

Proof. It is a direct consequence of Theorem 2 that D_k can be decomposed into directed 4-cycle and Lemma 1 that $D_{|V(G)|,k}$ can be decomposed into directed 4-cycles. □

3. Packing $D_t - P$ with Directed 4-Cycles

Before we give the proof of the first main theorem of this paper, we need a couple of important facts. For convenience, we use $k\ \vec{C}_t$ to denote k vertex-disjoint directed t-cycles.

Lemma 3. *D_4, $D_4 \setminus \vec{C}_4$, $D_6 \setminus 2\ \vec{C}_3$ and $D_7 \setminus 2\ \vec{C}_3$ have no $\vec{C}_4$-decompositions.*

Proof. The first two are easy to see and we prove the other two. Let D_6 be defined on Z_6 and $(0, 1, 2)$ and $(3, 4, 5)$ are the directed 3-cycles which are missing. Let $A = \{0, 1, 2\}$ and $B = \{3, 4, 5\}$. Clearly, D_6 can be written as $D_3 \cup D_{3,3} \cup D_3$ where the first D_3 is defined on A and the second one is defined on B. Now, we plan to decompose $(0, 2, 1) \cup D_{3,3} \cup (3, 5, 4)$ into directed 4-cycles. In order to use up the arcs in (0,2,1) and (3,5,4) respectively, each time we need two arcs from $D_{3,3}$, one in (A, B) and the other one in (B, A). For convenience, we call a directed 4-cycle obtained this way a match-up. This implies that we shall have 3 match-ups in order to use up all the arcs in (0,2,1)U(3,5,4). However, in total we have 9 arcs in (A, B) and 9 arcs in (B, A) respectively. So we shall have an odd number of match-ups to use all the arcs in $(0, 2, 1) \cup (3, 5, 4)$, it is either 5 match-ups or 3 match-ups. By direct constructions, it is not possible for 5 match-ups in this case. We see that all of them will leave at least two 2-cycles there. Therefore, a $\vec{C}_4$-decomposition of $D_6 \setminus 2\ \vec{C}_3$ is not possible. Since $D_7 \setminus 2\ \vec{C}_3 = (D_6 \setminus 2\ \vec{C}_3) \cup D_{1,6}$, then $D_7 \setminus 2\ \vec{C}_3$ has at least two 2-cycles left by the result for $D_6 \setminus 2\ \vec{C}_3$ and direct construction. □

Lemma 4. *$D_4 \setminus 2\ \vec{C}_2$, D_5, $D_5 \setminus \vec{C}_4$ and $D_5 \setminus 2\ \vec{C}_2$ have $\vec{C}_4$-decompositions.*

Proof. Let D_4 be defined on Z_4 and (0,2), (1,3) are the directed 2-cycles which are missing. Then $D_4 \setminus 2\ \vec{C}_2 = \{(0, 1, 2, 3), (3, 2, 1, 0)\}$. Let D_5 be defined on Z_5 and (0,2), (1,3) be the directed 2-cycles which are missing. Then $D_5 \setminus 2\ \vec{C}_2 = \{(4, 1, 0, 3), (4, 3, 2, 1), (4, 0, 1, 2), (4, 2, 3, 0)\}$. Let D_5 be defined on Z_5. Then $D_5 = \{(i, 1 + i, 3 + i, 2 + i | i \in Z_5)\}$ and $D_5 \setminus \vec{C}_4$ can be easily obtained. □

Lemma 5. *$D_6 \setminus \vec{C}_2$, $D_6 \setminus (\vec{C}_4 \cup \vec{C}_2)$, $D_6 \setminus 3\ \vec{C}_2$ and $D_6 \setminus \vec{C}_6$ have $\vec{C}_4$-decompositions.*

Proof. Since $D_6 \setminus 3\ \vec{C}_2 = (D_4 \setminus 2\ \vec{C}_2) \cup D_{2,4}$, we can get the result by Lemma 1 and Lemma 4. Let D_6 be defined on Z_6 where (5,4) and (3,2,1,0) are the missing cycles.

Then $D_6 \setminus (\vec{C}_2 \cup \vec{C}_4) = \{(4, 0, 1, 3), (5, 1, 2, 0), (5, 0, 4, 1), (4, 2, 3, 1), (5, 3, 0, 2), (5, 2, 4, 3)\}$. If we add (3,2,1,0) to the 4-cycles set, we get $D_6 \setminus \vec{C}_2$. If (0,1,2,3,4,5) is the missing cycle, then $D_6 \setminus \vec{C}_6 = \{(0, 3, 1, 4), (4, 1, 3, 0), (0, 2, 5, 1), (4, 3, 5, 2), (0, 5, 3, 2), (2, 1, 5, 4)\}$. □

Lemma 6. *$D_7 \setminus \vec{C}_2$, $D_7 \setminus 3\vec{C}_2$, $D_7 \setminus (\vec{C}_4 \cup \vec{C}_2)$, $D_7 \setminus \vec{C}_6$, $D_8 \setminus 2\vec{C}_2$, $D_8 \setminus \vec{C}_4$, $D_8 \setminus 2\vec{C}_4$, $D_8 \setminus (\vec{C}_5 \cup \vec{C}_3)$, $D_8 \setminus \vec{C}_8$, $D_8 \setminus 4\vec{C}_2$, $D_8 \setminus (2\vec{C}_2 \cup \vec{C}_4)$ and $D_8 \setminus (\vec{C}_6 \cup \vec{C}_2)$ have $\vec{C}_4$-decompositions.*

Proof. Since $D_7 \setminus \vec{C}_2 = D_5 \cup D_{2,5}$, $D_7 \setminus 3\vec{C}_2 = (D_5 \setminus 2\vec{C}_2) \cup D_{2,5}$, $D_7 \setminus (\vec{C}_4 \cup \vec{C}_2) = (D_5 \setminus \vec{C}_4) \cup D_{2,5}$, $D_8 \setminus 2\vec{C}_2 = (D_6 \setminus \vec{C}_2) \cup D_{2,6}$, $D_8 \setminus 4\vec{C}_2 = (D_4 \setminus 2\vec{C}_2) \cup D_{4,4} \cup (D_4 \setminus 2\vec{C}_2)$, $D_8 \setminus \vec{C}_4 \cup 2\vec{C}_2 = (D_4 \setminus 2\vec{C}_2) \cup D_{3,4} + (D_5 \setminus C_4)$, $D_8 \setminus \vec{C}_6 \cup \vec{C}_2 = (D_6 \setminus \vec{C}_6) \cup D_{2,6}$, it is easy to decompose them into directed 4-cycles by applying Lemma 4 and Lemma 5. By Theorem 2, D_8 can be decomposed into directed 4-cycles, so we delete a directed 4-cycle from it to get $D_8 \setminus \vec{C}_4$ and delete two vertex-disjoint directed 4-cycles from it to get $D_8 \setminus 2\vec{C}_4$. Let D_8 be defined on $Z_4 \cup \{a, b, c, d\}$ where $(0, 1, 2, 3, d)$ and (a, b, c) are the missing cycles. Then $D_8 \setminus (\vec{C}_5 \cup \vec{C}_3) = \{(a, 2, 0, 3), (1, a, 0, 2), (a, 3, 1, 0), (c, 2, b, 1), (d, 1, 3, 2), (d, 2, c, 1), (d, b, 2, a), (d, a, 1, b), (3, 0, c, d), (d, c, b, 0), (3, c, 0, b), (3, b, a, c)\}$. Let D_7 be defined on $Z_6 \cup \{\infty\}$ and let $(0, 1, 2, 3, 4, 5)$ be the missing cycle. Then $D_7 \setminus \vec{C}_6 = \{(\infty, 3, 1, 4), (\infty, 4, 0, 3), (\infty, 1, 3, 0), (\infty, 0, 4, 1), (0, 2, 5, 1), (4, 3, 5, 2), (\infty, 5, 3, 2), (\infty, 2, 0, 5), (2, 1, 5, 4)\}$. Let D_8 be defined on Z_8 and let $(0, 1, 2, 3, 4, 5, 6, 7)$ be the missing cycle. Then $D_8 \setminus \vec{C}_8 = \{(7, 4, 3, 0), (3, 2, 1, 0), (2, 4, 6, 5), (2, 5, 1, 4), (2, 7, 1, 6), (3, 1, 5, 7), (3, 7, 6, 1), (0, 2, 6, 4), (0, 4, 7, 2), (1, 7, 5, 4)\} + D_{\{0,3\},\{5,6\}}$. □

Lemma 7. *$D_8 \setminus (2\vec{C}_3 \cup \vec{C}_2)$, $D_9 \setminus (2\vec{C}_3 \cup \vec{C}_2)$, $D_9 \setminus 2\vec{C}_2$, $D_9 \setminus \vec{C}_4$, $D_9 \setminus (2\vec{C}_2 \cup \vec{C}_4)$, $D_9 \setminus (\vec{C}_6 \cup \vec{C}_2)$, $D_9 \setminus 2\vec{C}_4$, $D_9 \setminus (\vec{C}_3 \cup \vec{C}_5)$ and $D_9 \setminus \vec{C}_8$ have $\vec{C}_4$-decompositions.*

Proof. Let D_8 be defined on Z_8 where $(0, 1, 2)$, $(3, 4, 5)$ and $(6, 7)$ are the missing cycles. Then $D_8 \setminus (2\vec{C}_3 \cup \vec{C}_2) = \{(2, 1, 0, 3), (3, 0, 5, 4), (1, 3, 5, 6), (4, 0, 2, 6)^*, (7, 2, 3, 1)^*, (7, 5, 0, 4), (0, 7, 1, 6), (0, 6, 2, 7), (3, 7, 4, 6), (3, 6, 5, 7), (1, 4, 2, 5)^*, (5, 2, 4, 1)^*\}$. Since $D_9 \setminus (\vec{C}_2 \cup 2\vec{C}_3) = [D_8 \setminus (2\vec{C}_3 \cup \vec{C}_2] \cup D_{1,8}$, we can apply the decomposition obtained above to find a $\vec{C}_4$-decomposition of $D_9 \setminus (\vec{C}_2 \cup 2\vec{C}_3)$ by matching the arcs in $D_{1,8}$ with the directed 4-cycles in $D_8 \setminus (\vec{C}_2 \cup 2\vec{C}_3)$ which are marked with a "$*$". Here are the constructions: $(5, 2, 4, 1) \cup D_{\{\infty\},\{1,2\}} = \{(\infty, 2, 4, 1), (\infty, 1, 5, 2)\}$. $(1, 4, 2, 5) \cup D_{\{\infty\},\{4,5\}} = \{(\infty, 4, 2, 5), (\infty, 5, 1, 4)\}$. $(4, 0, 2, 6) \cup D_{\{\infty\},\{0,6\}} = \{(\infty, 0, 2, 6), (\infty, 6, 4, 0)\}$. $(7, 2, 3, 1) \cup D_{\{\infty\},\{3,7\}} =$

$\{(\infty, 3, 1, 7), (\infty, 7, 2, 3)\}$. This concludes the proof of this case. As for $D_9\setminus \vec{C}_8$, it is a special case of the proof of Theorem 4 Case (i). The remaining cases can be settled as those in Lemma 6. □

Lemma 8. *$D_{10}\setminus \vec{C}_2$, $D_{10}\setminus \vec{C}_6$, $D_{10} \setminus 3\vec{C}_2$, $D_{10} \setminus 2\vec{C}_3$, $D_{10} \setminus (\vec{C}_4 \cup \vec{C}_2)$, $D_{10} \setminus (\vec{C}_4 \cup \vec{C}_6)$, $D_{10} \setminus (\vec{C}_2 \cup \vec{C}_8)$, $D_{10} \setminus (\vec{C}_3 \cup \vec{C}_7)$, $D_{10} \setminus 2\vec{C}_5$, $D_{10} \setminus (2\vec{C}_3 \cup \vec{C}_4)$ and $D_{10}\setminus \vec{C}_{10}$ have $\vec{C}_4$-decompositions.*

Proof. Let D_{10} be defined on $Z_7 \cup \{x, y, z\}$ where $(0, 1, 2, 3, 4, 5, 6)$ and (x, y, z) are the missing cycles. Then $D_{10} \setminus (\vec{C}_3 \cup \vec{C}_7) = \{(0, 2, 5, 1), (4, 3, 6, 2), (0, 6, 3, 2), (2, 1, 5, 4), (0, 3, 1, 4), (4, 1, 3, 0), (6, 1, z, 4), (6, 4, y, 1), (x, 0, y, 3), (5, 0, x, 3), (5, 3, z, 0), (6, 5, z, y), (2, 6, y, x), (5, 2, x, z), (5, x, 1, y), (5, y, 4, x), (6, z, 1, x), (6, x, 4, z), (2, y, 0, z), (2, z, 3, y)\}$. For $D_{10}\setminus 2\vec{C}_5$, let D_{10} be defined on $V = A\cup B$ where $A = \{1, 2, a, b, c, d\}$ and $B = \{3, 4, 5, e\}$, $\vec{C}_5 = (1, 2, 5, 4, 3)$ and $\vec{C}_5 = (a, b, c, d, e)$. Let $\alpha = \{(a, 3, 5, 2), (e, 3, 4, 5), (e, 5, 3, 2), (3, e, 4, d), (e, d, 1, 4)\}$ and $\beta = \{(a, e, 1, 5), (d, 4, a, 5), (3, a, 4, 2), (3, d, 5, 1), (1, e, 2, 4), (b, e, c, 3), (e, b, 3, c), (b, 4, c, 5), (4, b, 5, c)\}$. Then add α to use up all arcs in B and leave a $\vec{C}_6 = (a, b, c, d, 1, 2)$ in A. The rest of the directed edges of A can be partitioned into 4-cycles by using Lemma 5. Now, it is left to complete the partition of $D_{A,B}$ into directed 4-cycles β. Let D_{10} be defined on Z_{10} and let $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ be the missing cycle. We can add two directed 4-cycles $(0, 9, 8, 7)$ and $(2, 5, 4, 3)$ to $\vec{C}_{10}$ and divide $\vec{C}_{10}$ into seven cycles $(0, 1, 2, 5, 6, 7)$, $(3, 4) \cup (8, 9)$, $(4, 5)$, $(0, 9)$, $(7, 8)$ and $(2, 3)$. Then $D_{10}\setminus\vec{C}_{10} = \{(0, 9, 8, 7), (2, 5, 4, 3), (0, 5, 1, 6), (6, 1, 5, 0), (4, 0, 2, 7), (4, 7, 1, 0), (9, 5, 7, 2), (9, 2, 6, 5), (3, 0, 7, 5), (8, 5, 2, 0), (8, 0, 3, 5), (2, 1, 7, 6), (9, 3, 8, 4), (4, 8, 3, 9)\} \cup D_{\{3,9\},\{1,6,7\}} \cup D_{\{4,8\},\{1,2,6\}}$. For $D_{10}\setminus 2\vec{C}_3$ and $D_{10}\setminus(2\vec{C}_3 \cup \vec{C}_4)$, they are the special cases of Theorem 4. Case (iii). As for other cases, they can be proved as those in Lemma 6. □

Lemma 9. *For each integer t, $t \geq 3$, $D_{2t} - \vec{C}_{2t}$ has a $\vec{C}_4$-decomposition.*

Proof. The proof is by induction. By Lemma 5, Lemma 6 and Lemma 8, it is true for $t = 3, 4, 5$. Assume the assertion is true for all orders less than t, we shall prove that the assertion is true for t.

Let D_{2t} be defined on $V = A \cup B \cup M$, $M = Z_{2t-10}$, $B = \{b_i | i \in Z_4\}$ and $A = \{a_i | i \in Z_6\}$. Let $\vec{C}_{2t} = (0, 1, \ldots, 2t - 12, 2t - 11, b_0, b_1, a_0, a_1, \ldots, a_5, b_2, b_3)$. Let $\alpha = \{(a_0, b_1, b_0, a_5), (a_5, b_0, 2t-11, b_2), (2t-11, 0, b_3, b_2)\}$ and $\beta = \{(b_0, b_2, b_1, b_3), (b_3, b_1, b_2, b_0)\}$. Add α to $\vec{C}_{2t}$. The union of the above three cycles can be partitioned into the following directed cycles $\vec{C}_{2t-10} = (0, 1, \ldots, 2t - 12, 2t - 11)$, $\vec{C}_6 = (a_0, a_1, \ldots, a_5)$, $\vec{C}_2 \cup \vec{C}_2 = (b_0, b_1) \cup (b_2, b_3)$, $(0, b_3)$, (a_0, b_1), $(2t - 11, b_2)$,

$(2t-11, b_0)$, (a_5, b_2) and (a_5, b_0). Then

$$\begin{aligned}
D_{2t}\setminus \vec{C}_{2t} &= D_{2t}\setminus(\vec{C}_{2t}\cup\alpha)\cup\alpha \\
&= [(D_M\setminus \vec{C}_{2t-10})\cup\alpha]\cup D_{M,A\cup B}\cup[D_{A\cup B}\setminus(\vec{C}_6\cup\vec{C}_2\cup\vec{C}_2)]\setminus[(b_0,a_5) \\
&\quad \cup(b_0,2t-11)\cup(b_2,a_5)\cup(b_2,2t-11)\cup(0,b_3)\cup(b_1,a_0)] \\
&= [(D_M\setminus \vec{C}_{2t-10})\cup\alpha]\cup D_{M,A\cup B}\cup(D_A\setminus\vec{C}_6)\cup D_B\setminus(\vec{C}_2\cup\vec{C}_2) \\
&\quad \cup D_{A,B}\setminus[(b_0,a_5)\cup(b_0,2t-11)\cup(b_2,a_5)\cup(b_2,2t-11) \\
&\qquad \cup(0,b_3)\cup(b_1,a_0)] \\
&= [(D_M\setminus \vec{C}_{2t-10})\cup\alpha\cup D_B\setminus(\vec{C}_2\cup\vec{C}_2)]\cup D_{B,A\cup M}\cup D_{A,M} \\
&\quad \cup(D_A\setminus\vec{C}_6)\setminus[(b_0,a_5)\cup(b_0,2t-11)\cup(b_2,a_5)\cup(b_2,2t-11) \\
&\quad \cup(0,b_3)\cup(b_1,a_0)] \\
&= [(\mathrm{I})\cup D_{\{b_0,b_2\},A\cup M\setminus\{a_5,2t-11\}}]\cup D_{\{b_1,b_3\},A\cup M} \\
&\quad \cup D_{A,M}\setminus(0,b_3)\setminus(b_1,a_0)\cup(D_A\setminus\vec{C}_6) \\
&= (\mathrm{I}')\cup D_{\{b_1\},A\cup M\setminus\{a_0\}}\cup D_{\{b_3\},A\cup M\setminus\{0\}}\cup D_{A,M}\cup(D_A\setminus\vec{C}_6) \\
&= (\mathrm{I}')\cup[(D_A\setminus\vec{C}_6)\cup D_{\{b_3\},A}]\cup D_{\{b_1\},A\cup M\setminus\{a_0\}}\cup D_{\{b_3\},M\setminus\{0\}}\cup D_{A,M} \\
&= (\mathrm{I}')\cup(\mathrm{II})\cup D_{\{b_1\},(A\setminus\{a_0\})\cup M}\cup D_{\{b_3\},M\setminus\{0\}}\cup D_{A,M} \\
&= (\mathrm{I}')\cup(\mathrm{II})\cup D_{\{b_1\},A\setminus\{a_0\}}\cup D_{\{b_1\},M}\cup D_{\{b_3\},M\setminus\{0\}}\cup D_{A\setminus\{a_0\},M}\cup D_{\{a_0\},M} \\
&= (\mathrm{I}')\cup(\mathrm{II})\cup D_{\{b_1\},A\setminus\{a_0\}}\cup D_{\{b_1\},M}\cup D_{\{b_3\},M\setminus\{0\}}\cup D_{A\setminus\{a_0\},M\setminus\{0\}} \\
&\quad \cup D_{A\setminus\{a_0\},\{0\}}\cup D_{\{a_0\},M} \\
&= (\mathrm{I}')\cup(\mathrm{II})\cup[D_{M\setminus\{0\},A\cup\{b_3\}\setminus\{a_0\}}\cup D_{\{0,b_1\},A\setminus\{a_0\}}\cup D_{M,\{a_0,b_1\}}] \\
&= (\mathrm{I}')\cup(\mathrm{II})\cup(\mathrm{III})
\end{aligned}$$

where $(\mathrm{I})=(D_{2t-10}\setminus\vec{C}_{2t-10})\cup\alpha\cup(D_B-\vec{C}_2\cup\vec{C}_2)$, $(\mathrm{I}')=(\mathrm{I})\cup D_{\{b_0,b_2\},A\cup M\setminus\{a_5,2t-11\}}$, $(\mathrm{II}) = D_{b_3,A}\cup(D_A\setminus\vec{C}_6) = \{(b_3,a_0,a_3,a_1), (b_3,a_4,a_1,a_3), (a_0,a_2,a_5,a_1), (b_3,a_1,a_4,a_0), (b_3,a_3,a_0,a_4), (a_4,a_3,a_5,a_2), (a_0,a_5,a_3,a_2), (b_3,a_2,a_1,a_5), (b_3,a_5,a_4,a_2)\}$. $(\mathrm{III}) = D_{M\setminus\{0\},A\cup\{b_3\}\setminus\{a_0\}}\cup D_{\{0,b_1\},A\setminus\{a_0\}}\cup D_{M,\{a_0,b_1\}}$ □

With the above preparations, we are now in a position to prove the first main theorem of this paper. For convenience, we shall denote the complete digraph defined on A by D_A in what follows.

Theorem 4. *Let v be an integer, $v\geq 8$ and P be a vertex-disjoint union of directed cycles in D_v. Then $\vec{C}_4\,|D_v\setminus P$ if and only if $v(v-1)-|E(P)|\equiv 0 \pmod 4$.*

Proof. The necessity is obvious and we prove the sufficiency by induction on v. Note that $v(v-1)-(v-2)(v-3)$ is congruent to 2 modulo 4 while $v(v-1)-(v-4)(v-5)$ is congruent to 0 modulo 4. Therefore, the plan of our proof is reducing the order v by 2 or 4. By Lemma 6 and Lemma 7, we conclude that $\vec{C}_4\,|D_8\setminus P$ and thus $v=8$

is true. Assume the assertion is true for all orders smaller than v and we shall prove the assertion is also true for $D_v \setminus P$. Of course, if P contains no arcs, then D_v has a $\vec{C}_4$-decomposition by Theorem 2. Otherwise, P contains at least one directed cycle.

First, if P contains a $\vec{C}_2$, then $D_v - P = [D_{v-2} \setminus (P \setminus \vec{C}_2)] \cup D_{2,v-2}$. By the induction process, $\vec{C}_4 \mid D_{v-2} \setminus (P \setminus \vec{C}_2)$ while by Lemma 1, $\vec{C}_4 \mid D_{2,v-2}$. Hence we finish this case. So, in what follows, we consider the case where P contains directed cycles of length at least 3, that is,

(i) P contains directed cycles of length not less than 6;
(ii) P contains a directed 4-cycle;
(iii) P contains two vertex-disjoint directed 3-cycles;
(iv) P contains a directed 3-cycle and a directed 5-cycle;
(v) P contains only directed 5-cycles;

Case (i). Let D_v be defined on Z_v, $A = \{a_0, a_1, a_3, a_4\}$, $B' = \{a_{t-1}, a_5, a_2, x\}$ and $B = Z_v \setminus (A \cup B')$. Suppose P contains a t-cycle $\vec{C}_t = \{a_0, a_1, \ldots, a_{t-2}, a_{t-1}\}$ where $v > t \geq 6$. Add cycle set $\alpha = \{(a_0, a_2, a_5, a_1), (a_4, a_3, a_{t-1}, a_2), (a_0, a_{t-1}, a_3, a_2), (a_2, a_1, a_5, a_4), (x, a_1, a_{t-1}, a_4)^*, (x, a_0, a_5, a_3)^*\}$ where the 4-cycles marked by "$*$" have double direction and $x \notin V(\vec{C}_t)$. Then $D_{Z_v} \setminus P = D_{Z_v} \cup \alpha \cup \vec{C}_t \setminus (P \setminus \vec{C}_t \setminus \alpha) = D_{Z_v} \setminus [P \setminus \vec{C}_t \cup (a_2, a_5, \ldots, a_{t-1}) \cup (a_1, a_0) \cup (a_3, a_4) \cup D_{A,B'}] = (D_{Z_v \setminus A} \setminus P') \cup (D_{Z_v \setminus A, A} \setminus D_{A,B'}) \cup [D_A \setminus (a_1, a_0) \setminus (a_3, a_4)] = (D_{Z_v \setminus A} \setminus P') \cup D_{Z_v \setminus A \cup B', A} \cup [D_A \setminus (a_1, a_0) \setminus (a_3, a_4)] = (D_{Z_v \setminus A} \setminus P') \cup D_{A,B} \cup [D_A \setminus (a_1, a_0) \setminus (a_3, a_4)]$ where $P' = P \setminus \vec{C}_t \cup (a_2, a_5, a_6, \ldots, a_{t-2}, a_{t-1})$. By the induction process $\vec{C}_4 \mid D_{Z_v \setminus A} \setminus P'$ and by Lemma 1 and Lemma 4, we have $\vec{C}_4 \mid D_{A,B}$ and $\vec{C}_4 \mid D_A \setminus (\vec{C}_2 \cup \vec{C}_2)$ respectively. Therefore, $\vec{C}_4 \mid D_v \setminus P$. On the other hand, if $v = t$, then t must be even. By Lemma 9, we conclude the proof of this case.

Case (ii). Let $P = P' \cup (a_0, a_1, a_2, a_3)$ and $x \in V(P')$. Then $D_v \setminus P = (D_{v-4} \setminus P') \cup (D_5 \setminus (a_0, a_1, a_2, a_3)) \cup D_{v-5,4}$ where D_5 is defined on $\{a_0, a_1, a_2, a_3, x\}$. The proof follows easily.

Case (iii). Let P be defined on Z_v, (a, b, c) and (d, e, f) be in P. Let $x \in V \setminus \{a, b, c, d, e, f\}$ and $P' = P \setminus [(a, b, c) \cup (d, e, f)]$. Let $\beta = \{(c, b, a, d), (d, a, f, e), (d, b, x, c), (b, d, f, x), (e, a, c, x), (a, e, x, f)\}$. Add β to P to get P', $D_{3,4}$, (b, c), (e, f) and (a, d). Then $D_{Z_v} \setminus P = D_{Z_v} \setminus [P' \cup (a, b, c) \cup (d, e, f) \cup \beta] \cup \beta = D_{Z_v \setminus \{b,c,e,f\}} \setminus (P' \cup (b, c) \cup (e, f) \cup (a, d) \cup D_{\{a,d,x\},\{b,c,e,f\}}) \cup \beta \cup D_{Z_v \setminus \{b,c,e,f\},\{b,c,e,f\}} \cup D_{\{b,c,e,f\}} = [D_{Z_v \setminus \{b,c,e,f\}} \setminus (P' \cup (a, d))] \cup \beta \cup [D_{\{b,c,e,f\}} \setminus (b, c) \setminus (e, f)] \cup (D_{Z_v \setminus \{b,c,e,f\},\{b,c,e,f\}} \setminus D_{\{a,d,x\},\{b,c,e,f\}}) = [D_{Z_v \setminus \{b,c,e,f\}} \setminus (P' \cup (a, d))] \cup \beta \cup [D_{\{b,c,e,f\}} \setminus (b, c) \setminus (e, f)] \cup (D_{Z_v \setminus \{a,b,c,d,e,f,x\},\{b,c,e,f\}})$. By induction, $\vec{C}_4 \mid D_{Z_v \setminus \{b,c,e,f\}} \setminus (P' \cup (a, d))$ and by Lemma 4, $\vec{C}_4 \mid D_{\{b,c,e,f\}} \setminus (b, c) \setminus (e, f)$, we have the proof.

Case (iv). Let $P' = P \setminus \vec{C}_3 \cup \vec{C}_5$. Then $D_v \setminus P = (D_{v-8} \setminus P') \cup D_{8,v} \cup (D_8 \setminus \vec{C}_3 \cup \vec{C}_5)$. By Lemma 6 and the hypothesis, this case is proved.

Case (v). Let D_v be defined on Z_v and $P' = P \setminus \vec{C}_5 \setminus \vec{C}_5$. Then $D_v \setminus P = (D_{v-10} \setminus P') \cup D_{v-10,10} \cup (D_{10} \setminus \vec{C}_5 \setminus \vec{C}_5)$. By the assertion and Lemma 8, we finish this case and the proof of this theorem. □

4. Packing $D_t \cup P$ with Directed 4-Cycles

In this section, we need the following Lemma.

Lemma 10. *Let P be a vertex-disjoint union of directed cycles defined on V and all cycles in P have length not less than 3. Then for any $a \notin V$, $D_{\{a\},V} \cup 2P$ has a $\vec{C}_4$-decomposition.*

The proof can be deduced from the following example immediately: $D_{\{a\},\{0,1,2\}} \cup (0,1,2) \cup (0,1,2) = D_{\{a\},\{0,1,2\}} \cup (0,1,2) \cup (0,1,2) = \{(a,0,1,2), (a,1,2,0), (a,2,0,1)\}$.

Lemma 11. *$D_4 \cup \vec{C}_4$, $D_6 \cup \vec{C}_2$ and $D_7 \cup \vec{C}_2$ have no $\vec{C}_4$-decompositions.*

Proof. The first can be easily seen. For the second, we have $D_6 \cup \vec{C}_2 = D_4 \cup D_{2,4} \cup (D_2 \cup \vec{C}_2)$ where $D_2 \cup \vec{C}_2 = 2\vec{C}_2$. By direct construction, we know there exist at least two double edges left. $D_7 \cup \vec{C}_2 = D_5 \cup D_{2,5} \cup (D_2 \cup \vec{C}_2)$. By direct construction, we know that there exist at least two 2-cycles left. □

Lemma 12. *$D_4 \cup 2\vec{C}_2$, $D_5 \cup 2\vec{C}_2$, $D_5 \cup \vec{C}_4$, $D_6 \cup \vec{C}_6$ and $D_6 \cup 3\vec{C}_2$ have $\vec{C}_4$-decompositions.*

Proof. Let D_4 be defined on Z_4 where $2\vec{C}_2 = (0,2) \cup (1,3)$. Then $D_4 \cup 2\vec{C}_2 = \{(0,1,3,2)^1, (1,0,2,3)^2, (0,2,1,3), (2,0,3,1)\}$. Let D_5 be defined on Z_5. Since $D_5 \cup 2\vec{C}_2 = (D_4 \cup 2\vec{C}_2) \cup D_{1,4}$, choose the 4-cycles marked by "1" and "2" respectively from above and associate them with $D_{\{4\},\{0,3\}}$ and $D_{\{4\},\{1,2\}}$ respectively. Since $D_{\{4\},\{0,3\}} \cup (0,1,3,2) = (4,0,1,3) \cup (3,2,0,4)$, $D_{\{4\},\{1,2\}} \cup (1,0,2,3) = (4,1,0,2) \cup (4,2,3,1)$. We have the decomposition. Thus $\vec{C}_4 \mid D_5 \cup 2\vec{C}_2$. Similarly, $D_5 \cup \vec{C}_4 = \{(i, 1+i, 3+i, 2+i) | i \in Z_5\} \cup \vec{C}_4$. Let D_6 be defined on Z_6 and $C_6 = (0,1,2,3,4,5)$. Then $D_6 \cup \vec{C}_6 = (D_6 \setminus (1,4)) \cup (0,1,4,5) \cup (1,2,3,4)$. Let D_6 be defined on Z_6 while $3\vec{C}_2 = (0,1) \cup (2,3) \cup (4,5)$. $D_6 \cup 3\vec{C}_2 = \{(5,1,2,3), (5,3,0,1), (4,0,1,3), (5,0,2,4), (3,2,1,0), (4,2,3,1), (5,4,3,2), (5,2,0,4), (5,4,1,0)\}$. □

Lemma 13. *$D_8 \cup \vec{C}_3 \cup \vec{C}_5$, $D_{10} \cup 2\vec{C}_5$ and $D_{10} \cup \vec{C}_3 \cup \vec{C}_7$ have $\vec{C}_4$-decompositions.*

Proof. Let D_8 be defined on Z_8, while (0, 1, 2) and (3, 4, 5, 6, 7) are the added cycles. Then $D_8 \cup \vec{C}_3 \cup \vec{C}_5 = \{(0, 3, 4, 2), (2, 4, 7, 1), (3, 0, 1, 7), (4, 5, 6, 7), (5, 7, 6, 1), (6, 5, 4, 1), (1, 2, 7, 5), (4, 6, 2, 1), (5, 2, 6, 4), (2, 5, 6, 7)\} \cup D_{\{0,3\},\{1,2,4,5,6,7\}}$. Let D_{10} be defined on Z_{10}, where (0, 1, 2, 3, 4) and (5, 6, 7, 8, 9) are the added cycles. Then $D_{10} \cup 2\vec{C}_5 = \{(7, 0, 2, 6), (7, 6, 2, 0), (8, 2, 9, 0), (8, 0, 9, 2), (0, 3, 5, 4), (7, 3, 0, 4), (7, 4, 5, 3), (3, 1, 6, 4), (4, 6, 1, 3), (8, 6, 3, 9), (5, 6, 8, 9), (5, 9, 3, 6), (7, 2, 1, 9), (8, 1, 2, 7), (8, 7, 9, 1), (6, 7, 8, 9), (1, 2, 3, 4), (0, 1, 5, 6), (4, 0, 6, 9), (9, 5, 1, 4)\} \cup D_{\{1,5\},\{0,7\}} \cup D_{\{3,4,5\},\{2,8\}}$. Let D_{10} be defined on Z_{10} where (0,1,2) and (3,4,5,6,7,8,9) are the added cycles. Then $D_{10} \cup \vec{C}_3 \cup \vec{C}_7 = \{(1, 2, 6, 4), (4, 6, 2, 1), (2, 7, 6, 9), (9, 6, 7, 2), (5, 6, 7, 8), (4, 5, 7, 8), (4, 5, 8, 9), (3, 0, 1, 9), (3, 4, 2, 0), (2, 4, 9, 1), (1, 6, 3, 7), (1, 7, 0, 6), (7, 4, 0, 9), (7, 9, 3, 4), (3, 5, 0, 8), (8, 0, 5, 3), (6, 0, 7, 3), (9, 0, 4, 3), (4, 8, 7, 5)\} + D_{\{5,8,0,3\},\{1,2\}} + D_{\{5,8\},\{6,9\}}$. □

Lemma 14. *$D_{10} \cup \vec{C}_2$, $D_{11} \cup \vec{C}_2$, $D_{10} \cup 3\vec{C}_2$, $D_{11} \cup 3\vec{C}_2$, $D_{10} \cup 5\vec{C}_2$ and $D_{11} \cup 5\vec{C}_2$, have $\vec{C}_4$-decompositions.*

Proof. Let D_{10} be defined on Z_{10} and $\vec{C}_2 = (8, 9)$. Then $D_{10} \cup \vec{C}_2 = \{(8, 3, 1, 0)^1, (8, 1, 3, 2)^2, (4, 5, 6, 7), (8, 7, 5, 4)^3, (8, 5, 7, 6)^4, (9, 1, 2, 3), (9, 3, 0, 1), (9, 8, 2, 0), (9, 0, 3, 8), (9, 8, 0, 2), (9, 2, 1, 8), (9, 4, 6, 5), (9, 5, 8, 4), (9, 6, 4, 7), (9, 7, 8, 6)^5\} \cup D_{\{0,1,2,3\},\{4,5,6,7\}}$. Let D_{11} be defined on $Z_{10} \cup \{\infty\}$ and associate 4-cycles marked by "1" with $D_{\{\infty\},\{3,0\}}$, $(8, 3, 1, 0)^1 \cup D_{\{\infty\},\{3,0\}} = (\infty, 3, 1, 0) \cup (\infty, 0, 8, 3)$. As before, associate 4-cycles marked by "2, 3, 4" and "5" with $D_{\{\infty\},\{1,2\}}$, $D_{\{\infty\},\{4,7\}}$, $D_{\{\infty\},\{5,6\}}$ and $D_{\{\infty\},\{8,9\}}$, respectively. Then, $\vec{C}_4 \mid D_{11} \cup \vec{C}_2$. Since $D_{10} \cup 3\vec{C}_2 = D_6 \cup 3\vec{C}_2 \cup D_{5,4} \cup D_5$, $D_{11} \cup 3\vec{C}_2 = D_6 \cup 3\vec{C}_2 \cup D_{5,6} \cup D_5$, $D_{10} \cup 5\vec{C}_2 = D_6 \cup 3\vec{C}_2 \cup D_{4,6} \cup (D_4 \cup 2\vec{C}_2)$, $D_{11} \cup 5\vec{C}_2 = D_6 \cup 3\vec{C}_2 \cup D_{5,6} \cup (D_5 \cup 2\vec{C}_2)$, these cases are proved. □

Lemma 15. *For integers t, l, $t \geq 3$, $l \geq 3$ and $l \equiv 0 \pmod 2$, $D_{2t} \cup \vec{C}_{2t}$ and $D_{2l} \cup l\vec{C}_2^{\,i}$ where $\vec{C}_2^{\,i} = (2i, 2i + 1)$ for $0 \leq i \leq l - 1$, have $\vec{C}_4$-decompositions.*

Proof. Let D_{2t} be defined on Z_{2t} and let $\vec{C}_{2t}^{\,i} = (0, 1, \ldots, 2t - 2, 2t - 1)$. Then $D_{2t} \cup \vec{C}_{2t} = \{(i, i + 1, 2t - 2 - i, 2t - 1 - i) | 0 \leq i \leq t - 2\} \cup [D_{2t} \setminus \sum_{i=0}^{t-3}(1 + i, 2t - 2 - i)]$. Let D_{2l} be defined on Z_{2l}. $A = \{(j, 1 + j, 2l - 2 - j, 2l - 1 - j) | 0 \leq j \leq l - 1, j \equiv 0 \pmod 2\}$ and $B = \{(j, 2l - 1 - j) | 0 \leq j \leq l - 1\}$. Then $D_{2l} \cup l\vec{C}_2^{\,i} = A \cup (D_{2l} \setminus B)$. By Theorem 4, we have the proof. □

Lemma 16. *For integers t_1, t_2, $t_1 \geq 4$, $t_2 \geq 4$, $D_{2t_1+2t_2+2} \cup \vec{C}_{2t_1+1} \cup \vec{C}_{2t_2+1}$ has $\vec{C}_4$-decompositions.*

Proof. Let $D_{2t_1+2t_2+2}$ be defined on V where $V = A \cup B, V_1 = V \backslash \{a_0, a_1, a_{2t_1}, b_0, b_1, b_{2t_2}\}$, $A = \{a_i | i \in Z_{2t_1+1}\}$ and $B = \{b_i | i \in Z_{2t_2+1}\}$. Let $\vec{C}_{2t_1+1} = (a_0, a_1, \ldots, a_{2t_1})$ and $\vec{C}_{2t_2+1} = (b_0, b_1, \ldots, b_{2t_2})$. We have $\vec{C}_{2t_1+1} = \{(a_{1+i}, a_{2+i}, a_{2t_1-1-i}, a_{2t_1-i}) | 0 \leq i \leq t_1 - 2\} \setminus \{(a_{1+i}, a_{2t_1-i}) | 0 \leq i \leq t_1 - 2\} \cup (a_0, a_1, a_{2t_1}) = A_1 \setminus A_2 \setminus (a_1, a_{2t_1}) \cup (a_0, a_1, a_{2t_1})$ where $A_1 = \{(a_{1+i}, a_{2+i}, a_{2t_1-1-i}, a_{2t_1-i}) | 0 \leq i \leq t_1 - 2\}$, $A_2 = \{(a_{1+i}, a_{2t_1-i}) | 1 \leq i \leq t_1 - 2\}$. Let $\vec{C}_{2t_2+1} = \{(b_{1+i}, b_{2+i}, b_{2t_2-1-i}, b_{2t_2-i}) | 0 \leq i \leq t_2 - 2\} - \{(b_{1+i}, b_{2t_2-i}) | 0 \leq i \leq t_2 - 2\} + (b_0, b_1, b_{2t_2}) = B_1 \setminus B_2 \setminus (b_1, b_{2t_1}) \cup (b_0, b_1, b_{2t_2})$ where $B_1 = \{(b_{1+i}, b_{2+i}, b_{2t_2-1-i}, b_{2t_2-i}) | 0 \leq i \leq t_2 - 2\}$, $B_2 = \{(b_{1+i}, b_{2t_2-i}) | 1 \leq i \leq t_2 - 2\}$.

Then $D_{2t_1+2t_2+2} \cup \vec{C}_{2t_1+1} \cup \vec{C}_{2t_2+1} = D_{2t_1+2t_2-4} \setminus (A_2 \cup B_2) \cup (A_1 \cup B_1) \cup [D_{6,2t_1 \cup 2t_2-4} \cup D_6 \setminus (a_1, a_{2t_1}) \setminus (b_1, b_{2t_2}) \cup (a_0, a_1, a_{2t_1}) \cup (b_0, b_1, b_{2t_2})] = (\mathrm{I}) \cup (\mathrm{II}) \cup (\mathrm{III})$.

$(\mathrm{I}) = D_{2t_1+2t_2-4} \setminus (A_2 \cup B_2)$, $(\mathrm{II}) = A_1 \cup B_1$,

$(\mathrm{III}) = D_{6,2t_1 \cup 2t_2-4} \cup D_6 \setminus (a_1, a_{2t_1}) \setminus (b_1, b_{2t_2}) \cup (a_0, a_1, a_{2t_1}) \cup (b_0, b_1, b_{2t_2}) = \{(D_{\{a_{2t_1}, b_0, b_1\}, V_1 \setminus \{e, f\}} \cup D_{\{a_0, a_1, b_{2t_2}\}, V_1}) + D_{\{a_1, b_0\}, \{b_1, b_{2t_2}\}} \cup \{(e, a_{2t_1}, f, b_1), (a_{2t_1}, b_0, e, b_1), (a_{2t_1}, b_1, f, b_0), (a_0, b_0, f, a_{2t_1}), (a_0, a_{2t_1}, e, b_0), (a_0, a_1, b_0, b_1), (a_0, a_1, a_{2t_1}, b_{2t_2}), (a_0, b_{2t_2}, b_0, a_1), (b_1, b_{2t_2}, a_{2t_1}, a_0)\}$ where $e, f \in V_1$. By Theorem 4, $\vec{C}_4 \mid (\mathrm{I})$, then this case is proved. □

With the above preparations, we are now in a position to prove the second main idea of this paper.

Theorem 5. *Let v be an integer, $v \geq 8$ and P be a vertex-disjoint union of directed cycles in D_v. Then $\vec{C}_4 | D_v \cup P$ if and only if $v(v-1) + |E(P)| \equiv 0 \pmod 4$.*

Proof. The necessity is obvious. We only need to prove the sufficiency.

We divide the proof into three cases.

Case (i). P contains even number of 2-cycles. Let $P_{2l} = P_{2l_1} \cup P_{2l_2}$ where $2l \leq v$, $l_2 \equiv 0 \pmod 2$, $P_{2l_2} = l_2 \vec{C}_2$ and all the cycles in P_{2l_1} have length longer than 3. Then $D_v \setminus P = D_{v-2l_2} \cup D_{2l_2, v-2l_2} \cup D_{2l_2} \cup P_{2l_1} \cup P_{2l_2} = D_{v-2l_2-2l_1} \cup D_{2l_1, v-2l_2-2l_1} \cup (D_{2l_1} \setminus P_{2l_1}) \cup (D_{1,2l_1} \cup 2P_{2l_1}) \cup [D_{2l_2, v-2l_2-2l_1} \cup D_{2l_2-1, 2l_1} \cup (D_{2l_2} \cup P_{2l_2})]$. By Lemma 1, Lemma 10, Lemma 15 and Theorem 4, we prove the cases.

Case (ii). P contains odd number of 2-cycles. Let $P = P_1 \cup \vec{C}_2$ or $P = P_2 \cup 3\vec{C}_2$ or $P = P_3 \cup 5\vec{C}_2$. For the first cases, we have $D_v \cup P = (D_{v-10} \cup P_1) \cup D_{10, v-10} \cup (D_{10} \cup \vec{C}_2)$. By Lemma 1, Lemma 14 and Case (i) of this section, we finish the proof. For the other cases, we can proceed similarly.

Case (iii). All cycles in P have length longer than 2.

If $v = 2k+1$, $P = P_{2l}$ where $l \leq k$, then we have $D_{2k+1} \cup P_{2l} = D_{2k-2l} \cup (D_{2l+1} \cup P_{2l}) \cup D_{2k-2l, 2l+1} = D_{2k-2l} \cup (D_{2l} - P_{2l}) \cup (D_{1,2l} + 2P_{2l}) \cup D_{2k-2l, 2l+1}$. By Lemma 1, Lemma 10 and Theorem 4, we get the proof.

If $v = 2k$, $P = P_{2l_1} \cup \vec{C}_{2l_2}$, then we have $D_{2k} \cup P = D_{2k-2l_2} \cup D_{2k-2l_2,2l_2} \cup D_{2l_2} \cup P_{2l_1} \cup \vec{C}_{2l_2} = (D_{2k-2l_2} \cup P_{2l_1}) \cup D_{2k-2l_2,2l_2} \cup (D_{2l_2} \cup \vec{C}_{2l_2}) = [D_{2k-2l_2-2l_1} \cup (D_{2l_1} \setminus P_{2l_1}) \cup D_{2k-2l_2-2l_1,2l_1} \cup (2P_{2l_1} \cup D_{1,2l_1})] \cup (D_{2k-2l_2-2l_1,2l_2} \cup D_{2l_1,2l_2-1}) \cup$ (III). By Lemma 1, Lemma 10, Lemma 15 and Theorem 4, we get the proof.

If $v = 2k$, $P = P_{2l_0} \cup \vec{C}_{2l_1+1} \cup \vec{C}_{2l_2+1}$, then we have $D_{2k} \cup P = (D_{2k-2l_1-2l_2-2} \cup P_{2l_0}) \cup D_{2k-2l_1-2l_2-2,2l_1+2l_2+2} \cup (D_{2l_1+2l_2+2} \cup \vec{C}_{2l_1+1} \cup \vec{C}_{2l_2+1}) = [D_{2k-2l_1-2l_2-2-2l_0} \cup (D_{2l_0} \setminus P_{2l_0}) \cup D_{2k-2l_1-2l_2-2-2l_0,2l_0} \cup (D_{1,2l_0} \cup 2P_{2l_0})] \cup (D_{2k-2l_1-2l_2-2-2l_0,2l_1+2l_2+2} \cup D_{2l_0,2l_1+2l_2+1}) \cup$(III). By Lemma 1, Lemma 10, Lemma 16 and Theorem 4, we get the proof.

Thus we conclude the proof of Theorem 5. □

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