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On latin cubes with prescribed intersections

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1. Introduction

A latin cube C of order v is a v-tuple $(L_1,L_2,...,L_v)$ of pairwise disjoint latin squares of order v. Let $C = (L_1,L_2,...,L_v)$ and $D = (M_1,M_2, \ldots ,M_v)$ be two latin cubes of order v (with the same entries), then the intersection of C and D is defined to be the number $|C \cap D| = \sum_{i=1}^{v} |L_i \cap M_i|$, where $|L_i \cap M_i|$ is the number of common entries of L_i and M_i. Moreover, we define $J[v]$ as the set of positive integers k such that there exist two latin cubes of order v with intersection k, and we define $I[v] = \{0,1,2,...,v^3-14\} \cup \{v^3-12,v^3-8,v^3\}$.

In [3] results on $J[v]$ were used in solving the intersection problem for Steiner quadruple systems of order $4v$, where v is the order of a Steiner quadruple system, and $v \geq 10$. Some of the results concerning $J[v]$ which were obtained in that paper are the following:

(1) $J[10] \supseteq I[10] \setminus \{10^3-21,10^3-14\}$.

(2) $J[v] \supseteq I[v] \setminus \{v^3-21,v^3-14\}$ for every even $v \geq 20$.

In this paper we prove that $J[v] = I[v]$ for every $v \geq 24$ and $J[v] \supseteq I[v] \setminus \{v^3-14\}$ when $20 \leq v \leq 23$.

2. Main theorems

It is easy to show that the intersection of two latin squares of order v cannot be v^2-5, v^2-3, v^2-2, and v^2-1. Hence we have the following lemma.

Lemma 2.1. *$J[v] \subseteq I[v]$ for every order v.*

Proof. It is well known that a latin cube is equivalent to a 3-quasigroup Q [2], and the set $\{(x,y,z) | (z,y,z \in Q\}$, with one component fixed, corresponds to a latin square. Since the intersections of two latin squares cannot be v^2-5, v^2-3, v^2-2, and v^2-1, we conclude that the intersections of two latin cubes of order v cannot be $v^3-13,v^3-11,v^3-10,v^3-9,v^3-7,...,v^3-1$. This implies that $J[v] \subseteq I[v]$.

Lemma 2.2. *$v^3-21 \in J[v]$ for every $v \geq 6$.*

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Proof. It is well known [1] that the partial latin square A of order 3 (Figure 2.1) can be embedded in a latin square $L = [\ell_{i,j}]$ of order $v \geq 6$. Let $M = [m_{i,j}]$ be a latin square of order v containing the subsquare B (Figure 2.1) in the upper-left corner. We construct a latin cube $C = (L_1, L_2, \ldots, L_v)$ by letting $L_1 = L$, $L_t = [\ell^{t}_{i,j}]$, $t = 2,3,\ldots,v$, $\ell^{t}_{i,j} = (\ell_{i,j})\alpha_t$, and $\alpha_t = \begin{pmatrix} m_{1,1} m_{1,2} \cdots m_{1,v} \\ m_{t,1} m_{t,2} \cdots m_{t,v} \end{pmatrix}$. It is easy to see that C is a latin cube which contains the partial latin cube D (Figure 2.2) in the upper-left corner of L_1, L_2, L_3. We can replace D by D' (Figure 2.2), and denote the new latin cube as C'. The theorem then follows as $|C \cap C'| = v^3 - 21$.

$A =$

1	2	
2	3	1
	1	3

$B =$

1	2	3
2	3	1
3	1	2

Figure 2.1

$D =$

1	2	
2	3	1
	1	3

2	3	
3	1	2
	2	1

3	1	
1	2	3
	3	2

$D' =$

2	1	
1	2	3
	3	1

3	2	
2	3	1
	1	2

1	3	
3	1	2
	2	3

Figure 2.2

Lemma 2.3. $v^3 - 14 \in J[v]$, $v \geq 24$.

Proof. Since the partial latin cube E (Figure 2.3) can be embedded in a latin cube L of order 12 (Figure 2.4), and a latin cube of order n can be embedded in a latin cube of order $m \geq 2n$ [3], then the partial latin cube E can be embedded in a latin cube of any order $v \geq 24$. We can replace E by E' (Figure 2.3), this concludes the proof.

$E =$

1	2	
2	1	3
	3	2

2	1	
1	3	2
	2	3

$E' =$

2	1	
1	3	2
	2	3

1	2	
2	1	3
	3	2

Figure 2.3

A_1

1	2	4	3	5	6
2	1	3	5	6	4
4	3	2	6	1	5
3	5	6	1	4	2
5	6	1	4	2	3
6	4	5	2	3	1

A_4

5	6	3	1	4	2
3	5	6	4	2	1
1	4	5	2	3	6
4	2	1	5	6	3
6	1	2	3	5	4
2	3	4	6	1	5

A_2

2	1	5	4	6	3
1	3	2	6	4	5
6	2	3	1	5	4
5	6	4	2	3	1
3	4	6	5	1	2
4	5	1	3	2	6

A_5

12	10	2	5	9	1
6	4	7	2	5	9
3	5	1	4	6	2
7	3	5	12	2	4
1	8	4	6	3	5
5	2	6	1	4	3

A_3

3	5	1	6	2	4
5	2	4	1	3	6
2	1	6	5	4	3
6	4	2	3	1	5
4	3	5	2	6	1
1	6	3	4	5	2

A_6

10	9	12	8	7	11
4	12	11	9	1	8
11	6	10	3	8	7
8	7	9	10	11	12
2	11	3	1	10	6
9	1	8	11	12	10

Figure 2.4

$$L = (L_1, L_2, \ldots, L_{12}),\ B_k(i,j) = \begin{cases} A_k(i,j)+n, & \text{if } A_k(i,j) \le n, \\ A_k(i,j)-n, & \text{if } A_k(i,j) > n \end{cases}.$$

$$L_k = \begin{array}{|c|c|} \hline A_k & B_k \\ \hline B_k & A_k \\ \hline \end{array}\ ,\ k = 1,2,3,4.$$

$$L_5 = \begin{array}{|c|c|} \hline A_5 & B_5 \\ \hline A_6 & B_6 \\ \hline \end{array}$$

$$L_6 \quad \begin{array}{|c|c|} \hline A_6 & B_6 \\ \hline A_5 & B_5 \\ \hline \end{array}$$

$$L_7 = \begin{array}{|c|c|} \hline B_5 & A_5 \\ \hline B_6 & A_6 \\ \hline \end{array}$$

$$L_8 = \begin{array}{|c|c|} \hline B_6 & A_6 \\ \hline B_5 & A_5 \\ \hline \end{array}$$

$$L_{k+8} \quad \begin{array}{|c|c|} \hline B_k & A_k \\ \hline A_k & B_k \\ \hline \end{array}\quad k = 1,2,3,4.$$

Figure 2.4 (continued)

Lemma 2.4. $J[10] \supseteq I[10] \backslash \{10^3-14\}$.

Proof. By Lemma 2.2 and the results obtained in [3].

For convenience of the following lemma, we denote the set $\{a+b \mid a \in A \text{ and } b \in B\}$ by $A + B$.

Lemma 2.5. $J[v] \supseteq I[v] \backslash \{v^3-14\}$ *for every* v, $20 \leq v \leq 39$.

Proof. Since a latin cube of order n can be embedded in a latin cube of order $m \geq 2n$ [3], let C be a latin cube of order v, $20 \leq v \leq 39$, containing a subcube B of order 10. B can, of course, be removed and replaced by any other latin cube on the same symbols. Now the following three parts of C can be permuted independently:

(1) the entries 1,2,...,10 in the right-lower corner of $L_1, L_2, \ldots, L_{10}$,

(2) the entries 1,2,...,10 but not in B or (1),

(3) the entries 11,12,...,v.

By applying the permutation to (1), (2), and (3) independently, we have $J[v] \supseteq J[10] + \{0, 10(v-10), 20(v-10), \ldots, 80(v-10), 100(v-10)\}$ $+ \{0, (v-10)v, 2(v-10)v, \ldots, 8(v-10)v, 10(v-10)v\}$ $+ \{0, v^2, 2v^2, \ldots, (v-12)v^2, (v-10)v^2\}$. Since $20 \leq v \leq 39$, it follows by Lemma 2.4 that $J[v] \supseteq I[v] \backslash \{v^3-14\}$.

Lemma 2.6. *If* $J[v] \supseteq I[v] \backslash \{v^3-14\}$, *then* $J[2v] \supseteq I[2v] \backslash \{(2v)^3-14\}$, *and* $J[2v+1] \supseteq I[2v+1] \backslash \{(2v+1^3)-14\}$, *for every* $v \geq 10$.

Proof. It is similar to Lemma 2.5.

Lemma 2.7. $J[v] \supseteq I[v] \backslash \{v^3-14\}$ *for every* $v \geq 20$.

Proof. By Lemma 2.5, and 2.6.

Now we have the following theorem.

Theorem 2.8. $J[v] = I[v]$ *for every* $v \geq 24$.

Proof. It is a direct result of Lemmas 2.3 and 2.7.

References

[1] T. Evans, Embedding incomplete latin squares, *Amer. Math. Monthly* 67 (1960), 958-961.

[2] A. Cruse, On the finite completion of partial latin cubes, *J. of Combinatorial Theory (A)* 17 (1974) 112-119.

[3] Hung-Lin Fu, Steiner quadruple systems of order $4v$ with prescribed intersections, *Ars Combinatoria* 21 (1986) 89-103.

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