DIRECTED TRIPLE SYSTEM HAVING A PRESCRIBED NUMBER OF TRIPLES IN COMMON

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Abstract.

In [1], Lindner and Rosa had shown that we can construct two Steiner triple systems of order v, briefly STS(v), with the common triples a prescribed number $k \in I_v = \{0, 1, 2, \cdots, t_v - 6, t_v - 4, t_v\}$, where $t_v = v(v-1)/6$ is the number of triples in any STS(v), and $v \geq 19$. Here we show that there exists a pair of directed triple system of order v [2], briefly DTS(v), having k (directed) triples in common for each $k \in I_v = \{0, 1, 2, \cdots, d_v - 2, d_v\}$ and $v \geq 4$, where d_v is the number of triples in any DTS(v).

1. Introduction.

By a DTS(v) we mean a pair (S, d), $|S| = v$ and d is a collection of ordered 3-subset (directed triple) of S in which every ordered pair of S belongs to exactly one triple of d and the triple $[a, b, c] \in d$ contains the ordered pairs (a, b), (b, c), and (a, c). If $v \equiv 1$ or 3 (mod 6), it is easy to see that we can construct a DTS(v) by substituting every triple $\{a, b, c\}$ in an STS(v) two directed triples $[a, b, c]$ and $[c, b, a]$. An STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6). A bit of reflection we see $d_v = 2t_v = v(v-1)/3$. It had been shown by Hung and Mendelsohn in [2] that the DTS(v) exists for all $v \equiv 0$ or 1 (mod 3) except 1. Before we go any further we introduce two construction-theorems.

THEOREM 1.1. *There exists a* DTS(v) *for each* $v = 3k$, $k \geq 3$.

PROOF. Let $S = \{x_i | x \in K, \ |K| = k$ and $i = 1, 2, 3\}$ and L_i be the corresponding idempotent latin square of $\{x_i | x \in K\}$ based on K. [3] Then (S, d) will be a DTS(v), $v = 3k$, where d is the collection of the following triples: (i) $[x_1, x_2, x_3]$ and $[x_3, x_2, x_1]$ for each $x \in K$, and (ii) $[x_1, y_2, z_1]$, $[x_2, y_3, z_2]$, $[x_3, y_1, z_3]$, if $x \neq y$ and z_i is the (x, y) entry of L_i, $i = 1, 2, 3$. In total, we have $2k + 3k(k-1)$ triples, which is $3k(3k-1)/3 = v(v-1)/3$. The proof is concluaed by checking every ordered paire of S occurs in at least one triple of d. We omit the detail here.

THEOREM 1.2. *There exists a* DTS(v) *for each* $v = 3k+1$, $k \geq 3$.

PROOF. Let $S^* = S \cup \{\infty\}$ and L_i be the corresponding idempotent latin square as above, then by choosing the following triples: (i) $[\infty, x_2, x_3]$ $[x_2, x_1, \infty]$, $[x_3, \infty, x_1]$, $[x_1, x_3, x_2]$ for each $x \in K$, and (ii) those triples of (ii) in theorem 1.1, we have a DTS(v), $v = 3k+1$. (By a similar argument as above.)

2. Main theorems.

It is not difficult to see if there are different idempotent latin squares (L_i) then we have different DTS(v). Moreover, the common entry outside the diagonal of two L_i's which correspond to the same set $\{x_i | x \in K\}$ produces a part of the common triples between two DTS(v).

Let $J[v] = \{k |$ there exists two DTS(v) which have k triples in common$\}$. Since it is impossible to have two directed triple systems which are different by exactly one triple, we have $J[v] \subseteq I_v$. Hence in order to prove $J[v] = I_v$, we need only show $J[v] \supseteq I_v$. We will use $(L_i, L_i') = m$ to denote L_i and L_i' based on the same set have m entries in common. It [3] Fu had shown that we can construct two idempotent latin squares of order v which have k entries in common for each $k \in \{v, v+1, \cdots, v^2-6, v^2-4, v^2\}$ and $v \geq 6$. This will be the main tool we use in proving two key lemmas.

LEMMA 2.1. $J[3] = \{0, 2\}$, $J[4] = \{0, 1, 2, 4\}$.

PROOF. It is clear if in DTS(3) we let $d_1 = \{[1, 2, 3], [3, 2, 1]\}$ and $d_2 = \{[1, 3, 2], [2, 3, 1]\}$, and in DTS(4) let $d_1 = \{[1, 2, 3], [2, 1, 4], [3, 4, 2], [4, 3, 1]\}$, $d_2 = \{[1, 3, 2], [2, 1, 4], [4, 2, 3], [3, 4, 1]\}$ and $d_3 = \{[1, 2, 4], [2, 1, 3], [3, 4, 2], [4, 3, 1]\}$.

LEMMA 2.2. *If* $v = 3k+1$, $k \geq 6$, *then* $J[v] = I_v$.

PROOF. Let (S^*, d_1) and (S^*, d_2) be two DTS(v) based on the same set as in theorem 1.2, the corresponding idempotent latin squares for (S^*, d_1) and (S^*, d_2) be L_i and L_i' respectively, $i = 1, 2, 3$. Moreover, we let $(L_i, L_i') = m_i$. It is easy to see, in this case, d_1 and d_2 have at least $m_1 - k + m_2 - k + m_3 - k$ triples in common. For each $x \in K$, $\{[x_2, x_1, \infty], [x_1, x_3, x_2], [\infty, x_2, x_3], [x_3, \infty, x_1]\}$ is a DTS(4), Hence we can pick one of the DTS(4) in lemma 2.1 for d_1 and the other for d_2. (or the same one) Consider 1 as x_1, 2 as x_2 and so on. This gives us the number of the possible common triples betweew d_1 and d_2: $\sum_{k \text{ copies}}\{0, 1, 2, 4\} + \sum_{i=1}^{3}\{(m_i - k)\}$, where the sum of two sets $A + B = \{a + b | a \in A \text{ and } b \in B\}$. Since $m_i \in \{k, k+1, \cdots, k^2 - 6, k^2 - 4, k^2\}$, we have $J[v] \supseteq I_v$ by simple computation.

LEMMA 2.3. *If* $v = 3k$, $k \geq 6$, *then* $J[v] \supseteq I_v - \{d_v - 3, d_v - 5\}$.

PROOF. Eor each $x \in K$, $\{[x_1, x_2, x_3], [x_3, x_2, x_1]\}$ is a DTS(3) and $\{[x_1, x_3, x_2], [x_2, x_3, x_1]\}$ is another DTS(3). By the same argument in lemma 2.2, $J[v] \supseteq \sum_{k \text{ copies}}\{0, 2\} + \sum_{i=1}^{3}\{m_i - k | m_i \in \{k, k+1, \cdots, k^2 - 6, k^2 - 4, k^2\}\}$ which is the set $I_v - \{d_v - 3, d_v - 5\}$.

In order to construct two DTS(v), $v = 3k$, which have $d_v - 3$, $d_v - 5$ entries in common, we need the following lemmas.

LEMMA 2.4. $J[v] = I_v$ *for* $v = 6, 7, 9, 10$.

PROOF. We mainly try to construct different DTS(v) by substituting a pair of triples of the form $[a, b, c]$, $[b, a, d]$ by $[b, a, c]$, $[a, b, d]$ $(c \neq d)$ or $[a, b, c]$, $[c, b, d]$ $(a = d$ or $a \neq d)$ by $[a, c, b]$, $[b, c, d]$ or $[c, a, b]$, $[d, b, a]$ by $[c, b, a]$, $[d, a, b]$, $(c \neq d)$ and a set of three triples of the form $[a, b, c]$, $[c, b, d]$, $[e, c, a]$ by $[c, a, b]$, $[b, c, d]$, $[e, a, c]$ or $[a, b, c]$, $[e, c, a]$, $[b, a, d]$, by $[b, c, a]$, $[a, b, d]$, $[e, a, c]$, then compute their intersections. Here we only list those DTS(v), $v = 6, 7, 9, 10$, and put the pair and the set of three triples together for sake of checking.

		$[a,$	$b,$	$c]$				
(i)	$v=6$	1	3	2		2	3	5
		6	3	4		5	4	3
		2	4	1		1	4	5
		5	6	2	—	6	5	1
		3	1	6				
		4	2	6				
(ii)	$v=7$	1	2	4		2	1	6
		4	5	7		5	4	2
		6	7	2		3	2	7
		7	1	3		1	7	5
		5	6	1		6	5	3
		4	3	6	—	3	4	1
		7	6	4				
		2	3	5				
(iii)	$v=9$	1	2	3		3	2	1
		4	5	6		6	5	4
		7	8	9		9	8	7
		1	5	9		9	5	1
		1	6	8		8	6	1
		1	4	7		7	4	1
		2	5	8		8	5	3
		3	6	9		9	6	2
		3	5	7		7	5	2
		2	6	7		7	6	3
		4	8	2	—	3	8	4
		9	4	3				
		2	4	9				

(iv)	$v=10$	1	5	3	3	5	1	|	1	6	2		2	6	1
		2	4	3	3	4	2	|	4	8	6		6	8	4
		5	7	6	6	7	5	|	4	9	5		5	9	4
		7	2	9	9	2	7		8	1	9		9	1	8
		7	3	8	8	3	7		2	5	8		5	2	10
		8	10	5	10	8	2		3	6	9		6	3	10
		9	10	6	10	9	3		1	4	7	—	7	10	4
		4	1	10					10	7	1				

LEMMA 2.5. $J[v] = I_v$, $v = 12, 13$.

PROOF. By using Fig. 1 (Appendix) as the corresponding idempotent latin square of order 4 in the construction-theorems, we can partition all the triples in (ii) into 12 classes, each has triples of the form $[a, d, b]$, $[b, d, c]$, $[c, d, a]$. We list them below (Appendix). We can substitute it by either $[a, d, c]$, $[c, d, b]$, $[b, d, a]$ or $[a, b, d]$, $[d, b, c]$, $[c, d, a]$ (also can use the same one), which gives the possible common triples 0, 1, or 3. Hence $J[12] \supseteq \sum_{4 \text{ copies}}\{0, 2\} + \sum_{12 \text{ copies}}\{0, 1, 3\} = I_{12}$ and $J[13] \supseteq \sum_{4 \text{ copies}}\{0, 1, 2, 4\} + \sum_{12 \text{ copies}}\{0, 1, 3\} = I_{13}$.

LEMMA 2.6. $J[15] = I_{15}$.

PROOF. An STS(7) (B, t_1) can be embedded in an STS(15) (S, t). [1] Hence the DTS(15) obtained by substituting every triple $\{a, b, c\} \in t$ two triples $[a, b, c]$, $[c, b, a]$ contains a subsystem (B, d_1) of order 7. Now we replace (B, d_1) by the DTS(7) in lemma 2.4, and use a similar computation we have)$J[15] = I_{15}$. (Since all the triples not in d_1 are paired off.)

LEMMA 2.7. $J[16] = I_{16}$.

PROOF. In [3], we also have the possible intersections between two idempotent latin squares of order 5 are: 5, 7, 9, 11, 13, 15, 17, 25. By lemma 2.2, $J[16] \supseteq \sum_{5 \text{ copies}}\{0, 1, 2, 4\} + \sum_{i=1}^{3}\{m_i - 5 \mid m_i \in \{5, 7, 9, 11, 13, 15, 17, 25\}\}$ which contains I_{16}.

LEMMA 2.8. $\{d_v - 5, d_v - 3\} \subseteq J[v]$ *for each* $v = 3k$, $k \geq 6$.

PROOF. We first claim two recursive constructions: (i) $k \in J[v]$ implies that $k + d_{2v+1} - d_v \in J[2v + 1]$, and (ii) $k \in J[v]$ implies that $k + d_{2v+4} - d_v \in J[2v + 4]$. In [2], we know a DTS($v$) can be embedded in a DTS($2v + 1$) (S, t). Let this DTS(v) be (B, d). Since $k \in J[v]$, there exists two DTS(v) (B, d_1) and (B, d_2) which have k triples in common. It is not difficult to see $(S, (t - d) \cup d_1)$ and $(S, (t - d) \cup d_2)$ are two DTS($2v + 1$) which have $k + d_{2v+1} - d_v$ triples in common. By a similar argument we also have (ii). By lemma 2.4 to lemma 2.7, $d_v - 5$, $d_v - 3$ are in $J[v]$ for each $v \in \{6, 7, 9, 10, 12, 13, 15, 16\}$. This implies $d_v - 5$ and $d_v - 3$ are in $J[v]$ recursiyely for each $v = 3k$, $k \geq 6$.

THEOREM 2.1. *$J[v] = I_v$ for each $v \equiv 0$ or $1 \pmod 3$, $v \geq 4$.*

PROOF. By lemma 2.1 through 2.8 we conclude the proof.

Appendix

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

Figure 1

$v = 12$ or 13

2	5	4	3	5	2	4	5	3
1	6	3	3	6	4	4	6	1
1	7	4	2	7	1	4	7	2
1	8	2	2	8	3	3	8	1
6	9	8	7	9	6	8	9	7
5	10	7	7	10	8	8	10	5
5	11	8	6	11	5	8	11	6
5	12	6	6	12	7	7	12	5
10	1	12	11	1	10	12	1	11
9	2	11	11	2	12	12	2	9
9	3	12	10	3	9	12	3	10
9	4	10	10	4	11	11	4	9

References

[1] C. C. LINDNER and A. ROSA, *Steiner triple systems having a prescribed number of triples in common*, Can. J. Math., Vol. 27, 1975, 1166–1175.

[2] S. H. Y. HUNG and N. S. MENDELSOHN, *Directed triple systems*, J. of Comb. Theory (A) **14**, 310–318 (1973).

[3] H. L. FU, *On the constructions of certain types of latin squares having prescribed intersections*, Dissertation of Ph. D. in Auburn University, Auburn, Alabama, U. S. A.

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