NYCU Probability Qualifying Exam (Spring 2026)

- There are six problems in total, please choose only five to answer.
- Each problem is worth 20 points.
- The passing grade is 60 points out of 100 total points.

1. An urn contains 2026 blue and 115 red balls. Balls are removed at random until the first blue ball is drawn. Find the expected number drawn. (Hint: the answer in fraction, not in decimal.)

2. Let X and Y be independent random variables, where X has an arc sine distribution and Y a Rayleigh distribution:
$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad |x| < 1, \qquad f_Y(y) = y\,\mathrm{e}^{-\frac{1}{2}y^2}, \qquad y > 0.$$
Find the distribution law of XY.

3. Let $U_1, U_2, \ldots, U_{2026}$ be independent random variables with the uniform distribution on $[0,1]$, and let
$$U_{(1)} = \min_{1\le r\le 2025} U_r, \qquad U_{(2025)} = \max_{1\le r\le 2025} U_r.$$
   - Find $\mathbb{E}(U_{(1)})$;
   - Find $\mathbb{P}\big(U_{(2025)} - U_{(1)} \le U_{2026}\big)$.

4. Let $G = (V, E)$ be a random graph with $m = |V|$ vertices and edge-set E. Write d_v for the degree of vertex v, that is, the number of edges meeting at v. Let Y be a uniformly chosen vertex, and Z a uniformly chosen neighbour of Y. Show that
$$\mathbb{E}(d_Z) \ge \mathbb{E}(d_Y).$$

5. Let $X = (X_n)_{n\ge 0}$ be a discrete-time Markov chain with finite or countable state space S and transition matrix $P = (P(i,j))_{i,j\in S}$. The chain is said to be *reversible in equilibrium* if there exists a probability distribution $\pi = (\pi(i))_{i\in S}$ on S such that:

   (1) π is a stationary distribution for P, that is,
$$\pi(j) = \sum_{i\in S} \pi(i)P(i,j) \qquad \text{for all } j \in S;$$

   (2) the *detailed balance equations* hold:
$$\pi(i)P(i,j) = \pi(j)P(j,i) \qquad \text{for all } i, j \in S.$$

In this case, the joint distribution of (X_n, X_{n+1}) under $X_n \sim \pi$ is symmetric in its two coordinates, and the Markov chain is said to be reversible with respect to π. Let X be a discrete-time Markov chain with state space $S = \{1, 2\}$, and transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

Charaterize those (α, β) such that the chain is reversible in equilibrium.

6. Let $X_1, X_2, \ldots$ be independent random variables with

$$X_n = \begin{cases} 1, & \text{with probability } (2n)^{-1}, \\ 0, & \text{with probability } 1 - n^{-1}, \\ -1, & \text{with probability } (2n)^{-1}. \end{cases}$$

Let $Y_1 = X_1$ and for $n \geq 2$

$$Y_n = \begin{cases} X_n, & \text{if } Y_{n-1} = 0, \\ nY_{n-1}X_n, & \text{if } Y_{n-1} \neq 0. \end{cases}$$

- Show that Y_n is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, Y_2, \ldots, Y_n)$.
- Show that Y_n does not converge almost surely.