

PhD Qualifying Exam in Numerical Analysis

Spring 2026

Instructions

- Choose exactly five (5) problems.
- Clearly indicate the five problems you wish to have graded (e.g., write down the problem numbers on the first page of the answer sheet).
- If more than five problems are attempted and no selection is indicated, only the first five problems in numerical order will be graded.
- Show all work clearly and logically. No credit will be given for answers without supporting justification.
- State any theorems you use and verify that their hypotheses are satisfied.
- Calculators are not allowed.
- No electronic devices are permitted unless explicitly authorized.

1. Given a linear system $Ax = b$ where a matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and b is a vector in \mathbb{R}^n , we use the stationary iterative method to numerically find the solution of the linear system. For that purpose, we decompose the matrix $A = M - N$ where a matrix $M \in \mathbb{R}^{n \times n}$ is nonsingular and N is a matrix in $\mathbb{R}^{n \times n}$. Consider the iteration method

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b,$$

with a given initial guess x^0 where $x^{(k)}$ are vectors in \mathbb{R}^n . Please answer the following questions:

- (a) (7 pts) Show that the necessary and sufficient condition on the convergence of the iterative method with any initial guess x^0 is

$$\rho(M^{-1}N) < 1,$$

where $\rho(M^{-1}N)$ is the spectral radius of the matrix $M^{-1}N$.

- (b) (6 pts) If $A = D - (L + U)$ where D is the diagonal part of the matrix A , L is the strictly lower triangular part of the matrix A with a minus sign and U is the strictly upper triangular part of the matrix A with a minus sign. Based on this formulation, please write down Jacobi and Gauss-Seidal methods for the linear system.

- (c) (7 pts) Show that the algorithm of the Gauss-Seidal method is convergent with any initial guess x^0 if the matrix A is symmetric and positive definite.

2. Given a function $f(x) = \frac{1}{x^2+1}$ defined on the interval $[-5, 5]$. Let $f_n(x)$ be a function that interpolates $f(x)$ by using Lagrange interpolation on equally spaced nodes $\Delta x = \frac{10}{n}$ where $n + 1$ is the number of nodes. Please answer the following questions:

(a) (7 pts) Show that for any $x \in [-5, 5]$, we have

$$f(x) - f_n(x) = w_n(x)f[x_0^{(n)}, \dots, x_n^{(n)}, x]$$

where $x_j^{(n)} = -5 + j\Delta x, j = 0, \dots, n$, $w_n(x) = \prod_{j=0}^n (x - x_j^{(n)})$ and $f[x_0^{(n)}, \dots, x_n^{(n)}, x]$ is the $(n + 1)$ st Newton divided difference of f .

(b) (7 pts) Show that

$$f[x_0^{(n)}, \dots, x_n^{(n)}, x] = \frac{1}{i w_n(i)} \frac{r_n}{x^2 + 1},$$

where i is the imaginary number and $r_n = x$ if n is even; $r_n = i$ if n is odd.

(c) (6 pts) Show that f_n does not converges to f at $x = 4$ as n goes to infinity.

3. Given $n + 1$ distinct points $a = x_0 < x_1 < \dots < x_n = b$ on the interval $[a, b]$ ($a < b$) and $f \in C^2([a, b])$, please answer the following questions:

(a) (10 pts) Show that there exists a unique piecewise-polynomial $S(x)$ with degree 3 such that $S(x) = S_j(x)$ for $x \in [x_j, x_{j+1}]$, $j = 0, \dots, n - 1$, satisfying

- (1) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for $j = 0, \dots, n - 1$;
- (2) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for $j = 0, \dots, n - 2$;
- (3) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for $j = 0, \dots, n - 2$;
- (4) $S''(x_0) = S''(x_n) = 0$.

(b) (10 pts) Show that $S(x)$ in part (a) satisfies

$$\int_a^b (S''(x))^2 dx \leq \int_a^b (f''(x))^2 dx$$

where the equality holds if and only if $f(x) = S(x)$ on $[a, b]$.

4. Given the ordinary differential equation $\frac{dy}{dt} = f(t, y(t))$ with the initial condition $y(0) = y_0$. Assume that f is continuous in t and Lipschitz in y with Lipschitz constant $L > 0$ on a region containing the exact and numerical solutions, and that the exact solution $y(t)$ is sufficiently smooth and given the time step size $h > 0$, please answer the following questions:

- (a) (6 pts) Derive explicit Euler method of the given ordinary differential equation.
- (b) (7 pts) compute the local truncation error of the explicit Euler method.
- (c) (7 pts) Show that the convergence rate of the explicit Euler method is of the first order.

5. Consider the two-point boundary value problem

$$(P1) \begin{cases} -u''(x) + u(x) = f(x), & x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

We can rewrite problem (P1) as the following weak formulation:

$$(P2) : \text{ Find } u \in V \text{ satisfying } a(u, v) = (f, v), \text{ for all } v \in V,$$

where the functional space $V = H_0^1(0, 1)$, $(f, v) = \int_0^1 f v dx$ denotes the scalar product of $L^2(0, 1)$ and the bilinear form $a(u, v) : V \times V \rightarrow \mathbb{R}$ is defined by

$$a(u, v) = \int_0^1 u'v' dx + \int_0^1 uv dx.$$

We use the Galerkin method to solve (P2) by introducing V_h as a finite dimensional vector subspace of V and approximate (P2) by the problem

$$(P3) : \text{ Find } u_h \in V_h \text{ satisfying } a(u_h, v_h) = (f, v_h), \text{ for all } v_h \in V_h,$$

Please answer the following questions:

- (a) (10 pts) If $f \in C^0([0, 1])$, show that $\|u\|_\infty \leq \frac{1}{2} \frac{e}{e-e^{-1}} \|f\|_\infty$.
(b) (10 pts) If u and u_h are the solutions of (P2) and (P3) respectively and $f \in L^2(0, 1)$, show that

$$|u - u_h|_{H^1(0,1)} \leq c \min_{w_h \in V_h} |u - w_h|_{H^1(0,1)},$$

where c is a positive constant independent of h and we endow the space $H_0^1(0, 1)$ with the following norm

$$|v|_{H^1(0,1)} = \left(\int_0^1 (v'(x))^2 dx \right)^{1/2}.$$

6. Assume that $u = u(x, t)$ satisfies the heat equation

$$(P4) : u_t - \nu u_{xx} = f, \quad (x, t) \in (0, 1) \times (0, \infty),$$

subject to the boundary condition

$$u(0, t) = u(1, t) = 0, \quad t > 0,$$

and the initial value condition

$$u(x, 0) = u_0(x), x \in [0, 1],$$

where $\nu > 0$ is the diffusive constant and $f(x, t)$ is the forcing term. For the solution $u(x, t)$ of (P4), we define the energy $E(t)$ on the time interval as

$$E(t) = \int_0^1 u^2(x, t) dx.$$

Now, consider the central difference for the spatial variable and θ method for the temporal variable and obtain the following numerical scheme

$$(P5) : \begin{cases} \frac{u_i^{k+1} - u_i^k}{\Delta t} - \nu \theta \frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{(\Delta x)^2} - \nu(1 - \theta) \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} \\ = \theta f_i^{k+1} + (1 - \theta) f_i^k, i = 1, \dots, n - 1; k = 0, 1, \dots; \\ u_0^k = u_n^k = 0, k = 0, 1, \dots; \\ u_i^0 = u_0(x_i), i = 1, \dots, n - 1, \end{cases}$$

where $\Delta x = \frac{1}{n}$, $\Delta t > 0$ and $\theta > 0$. Here, u_i^k is the approximation of u at $(x, t) = (i\Delta x, k\Delta t)$ and $f_i^k = f(i\Delta x, k\Delta t)$. We define $U^k = (u_1^k \quad u_2^k \quad \dots \quad u_{n-1}^k)^T$ and rewrite (P5) in terms of the matrix form as follows:

$$(P6) : U^{k+1} - \theta \Delta t A U^{k+1} = U^k + (1 - \theta) \Delta t A U^k + \theta \Delta t F^{k+1} + (1 - \theta) \Delta t F^k,$$

where A is some $(n - 1) \times (n - 1)$ matrix and F^k is related to the forcing term f^k . Please answer the following questions:

(a) (7 pts) Show that $E(t)$ satisfies the following inequality for $t \geq 0$,

$$E(t) \leq e^{-\gamma t}(E(0)) + \frac{1}{\gamma} \int_0^t e^{\gamma(s-t)} F(s) ds,$$

where $\gamma = \frac{\nu}{c_p^2}$, $F(t) = \int_0^1 f^2(x, t) dx$ and c_p is the Poincare constant.

(b) (7 pts) Show that the problem (P6) is solvable for any given U^k , $\Delta t > 0$ and $\theta > 0$.

(c) (6 pts) Find all values of θ to determine the numerical scheme (P5) is unconditionally stable.