NATIONAL YANG MING CHIAO TUNG UNIVERSITY

2025 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 113-2

1. (a) (10 %) Show that the solution of the initial value problem

$$y' = \sqrt{|y|}, \quad y(t_0) = 0 \tag{1}$$

exists, but is not unique by producing two linearly independent solutions.

(b) (5 %) Does this contradict the Fundamental Existence and Uniqueness theorem? Please state the fundamental existence and uniqueness theorem for ODE.

(c) (5 %) What can you say about the following perturbations of the above system:

$$y' = \sqrt{|y|} + 0.1, \quad y(t_0) = 0 \tag{2}$$

2. (20 %) Suppose that square matrix A has a negative eigenvalue. Show that the linear system

$$\dot{x} = Ax$$

has at least one nontrivial solution $x(t)$ that satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

3. (20 %) Use Lyapunov's method to study the stability of the zero solution for the planar system of equations

$$x' = y^2 - x^3, \quad y' = -y - 2xy.$$

4. Consider a Lotka-Volterra two species competition model

$$\begin{aligned} x_1' &= \gamma_1 x_1 (1 - \frac{x_1}{K_1}) - \alpha_1 x_1 x_2 \\ x_2' &= \gamma_2 x_2 (1 - \frac{x_2}{K_2}) - \alpha_2 x_1 x_2 \end{aligned} \tag{3}$$

where γ_1, γ_2, α_1, α_2, K_1, K_2 are positive constants and $\frac{\gamma_1}{\alpha_1} < K_1$, $\frac{\gamma_2}{\alpha_2} < K_2$.

(a) (5 %) Classify the equilibria

(b) (5 %) Sketch the phase portrait

(c) (5 %) Determine the stable and unstable manifolds around each saddle point

(d) (5 %) What can you say about the global asymptotic behavior of the nonlinear system?

5. Consider the system in polar coordinates

$$\begin{aligned} r' &= 2r^2 - r^3 + \mu r \\ \theta' &= 1 \end{aligned} \tag{4}$$

(a) (10 %) Identify all equilibria and limit cycles and classify their existence and stability as they depend on parameter μ

(b) (10 %) Sketch sample phase portraits for different values of μ