PhD Qualifying Exam in Numerical Analysis

Spring, 2025

1. Consider a linear system Ax=b by where A=$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (i) (4 pts) Construct the associated Krylov space given an initial guess $x_0 = [0\ 0\ 0\ 0]^T$.

   (ii) (8 pts) Solve the linear system by using the Arnoldi iteration.

   (iii) (8 pts) It can be shown that the number of Arnoldi iteration required equals 4 when $b = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Explain the reason why the number of iterations is less than 4 in (ii) by finding the minimal polynomial p, $p(0) = 1$, such that $p(A)b = 0$.

2. (20 pts) Suppose that f has a continuous second derivative on [0,1]. Show that there is a point $\eta \in (0,1)$ such that

$$\int_0^1 xf(x)dx = \frac{1}{2} f\left(\frac{2}{3}\right) + \frac{1}{72} f''(\eta).$$

Hint: using Gauss quadrature

$$\int_0^1 f(x)w(x)dx - \sum_{i=0}^{n} w_i f(x_i) = K_n f^{(2n+2)}(\eta),$$

here x_i and $w_i, i = 0 \cdots n,$ are quadrature points and weights, respectively.

3. (20 pts) Suppose a cubic spline $s(x)$ is constructed to approximate $f(x) = x^3$ for $x \in [0,1]$. Let $\sigma_i = s''(x_i), for\ i = 0 \cdots m,\ m \geq 2$. Show that $s(x) \neq f(x)$ if natural spline boundary conditions are applied. On the other hand, if $\sigma_0 = f''(0)$ and $\sigma_m = f''(1)$, then $s(x) = f(x)$.

4. (20 pts) Consider a model ordinary differential equation

$$\begin{cases} y' = \lambda y \\ y(0) = 1 \end{cases}$$

here $t \in (0, \infty)$ and $\lambda \in (-\infty, 0)$. Suppose the equation is numerically solved by the RK method defined by the following Butcher tableau

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

(i) (10 pts) Show that the numerical method is absolute stable when the time step h satisfies $-2.5 < h\lambda < 0$.

(ii) (10 pts) Check the consistency and determine the order of convergence of the method.

5. (20 pts) Consider the following boundary-value problem

$$\begin{cases} -\varepsilon u'' + u' = f(x), \quad \text{for } x \in [0,1] \\ u(0) = 0,\ u(1) = 1 \end{cases},$$

where ε is a constant and f is continuous on [0,1]. Suppose we solve the equation by the Galerkin finite element method where the solution is piecewise linear on a uniform mesh with mesh size $h = \frac{1}{n}$ for some $n \in N$. Let $u_h = (u_0, u_1 \quad \dots \quad u_n)^T$ be the finite element solution with $u_0 = 0, u_n = 1$. The finite element solution satisfies the discrete equation

$$A_h u_h = f_h$$

(i) (8 pts) Derive the matrix A_h and f_h after the boundary conditions are applied for the case n=3.

(ii) (6 pts) Prove that $\|(u - u_h)'\| \leq \gamma \|u''\|$, here $\|\cdot\|$ is the usual square integrable norm, i.e. the $L^2([0,1])$ norm.

(iii) (6 pts) For the case ε =0.1 and $f(x) = 0$, find the exact solution and compare with u_h with n=3. From your observation, do you think the error estimation is useful? Justify your answer.