NATIONAL YANG MING CHIAO TUNG UNIVERSITY

Real Analysis Ph.D. Qualifying Exam, Fall 2024

Throughout this exam, $m(E)$ denotes the Lebesgue measure of E.

1. Let $\{a_n\}_{n\geq 2}$ be a sequence of real numbers with $|a_n| < \log n$. Let
$$f(x) = \sum_{n=2}^{\infty} a_n n^{-x} \quad \text{for } x \geq 2.$$
   (a) (10%) Prove that $f \in L^1([2,\infty))$.
   (b) (6%) Prove that
$$\sum_{n=2}^{\infty} \int_2^{\infty} a_n n^{-x} dx = \int_2^{\infty} f(x) dx.$$

2. Let f and $\{f_n\}$ be measurable and finite almost everywhere in E.
   (a) (10%) If $f_n \to f$ almost everywhere on E and $m(E) < \infty$, show that for every $\varepsilon > 0$,
$$\lim_{n\to\infty} m(\{x \in E : |f(x) - f_n(x)| > \varepsilon\}) = 0.$$
   (b) (6%) Does the statement (a) remain true if the finiteness condition of $m(E)$ is removed? Why?
   (c) (10%) Is the statement (a) still valid if the condition is changed to $f_n \to f$ in $L^1(E)$? Why?

3. Let $f : [a,b] \to \mathbb{R}$ have bounded variation.
   (a) (8%) Show that f is equal to the difference of two increasing functions.
   (b) (8%) Show that $\int_a^b |f'(x)| dx \leq V[a,b]$, where $V[a,b]$ is the total variation of f on $[a,b]$.

4. (15%) Let f be measurable and periodic with period 1, this is, $f(t+1) = f(t)$. Suppose that there is a finite c such that
$$\int_0^1 |f(a+t) - f(b+t)| dt \leq c$$
for all a and b. Show that $f \in L^1([0,1])$.

5. (15%) Prove that the limit
$$\lim_{p\to 0^+} \left(\frac{1}{m(E)} \int_E f^p \right)^{1/p} = \exp\left(\frac{1}{m(E)} \int_E \log f \right),$$
where E be a measurable set in $\mathbb{R}^n$ with $m(E) < \infty$, and $f > 0$ almost everywhere in E with f and $\log f$ both in $\in L^1(E)$. (Hint: $(f^p - 1)/p \to \log f$ as $p \to 0^+$.)

6. (12%) Suppose $1 \leq p < r < q < \infty$. Prove that $L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \subseteq L^r(\mathbb{R}^n)$.