NATIONAL YANG MING CHIAO TUNG UNIVERSITY

2024 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 113-1, 2024

1. (20 %) Solve the following differential equations

$$\begin{cases} \frac{dx}{dt} = -y - t \\ \frac{dy}{dt} = x + t \\ x(0) = 1, \ y(0) = 0 \end{cases}$$

2. (20 %) Classify the equilibrium points(as sinks, sources or saddles) of the following Lorentz equations

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{pmatrix} \tag{1}$$

   where $\mu > 0$

3. Consider the equation $\frac{dx}{dt} = f(t, x)$. Assume that $|f(t, x)| \leq \phi(t)|x|$ for all t, x and $\int_{-\infty}^{\infty} |\phi(t)| dt < +\infty$.
   (a) (10 %) Show that every solution approach a constant as $t \to \pm\infty$.
   (b) (10 %) If, in addition,

   $$|f(t, x) - f(t, y)| \leq \phi(t)|x - y|$$

   for all x, y. Prove that there is a one-to-one correspondence between the initial values and the limit values of the solutions.

4. (20 %) Show that the system

$$\begin{cases} x' = y \\ y' = -x + y(1 - x^2 - 2y^2) \end{cases}$$

   is positively invariant in some annulus. Please find that annulus and show that there exits at least one periodic solution in that annulus.

5. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = 2x^2 y + 2xy^2, \\ \dfrac{dy}{dt} = -3x^3 - x^2 y - 8y, \quad (x, y) \in \mathbb{R}^2, \end{cases}$$

   (a) (10 %) Provide a definition of "Lyapunov function" for the autonomous system $d\mathbf{x}/dt = f(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2$, where the origin is an equilibrium point.
   (b) (10 %) Prove or disprove that the origin is asymptotically stable.