# PhD Qualifying Exam in Numerical Analysis

Fall, 2024

Please do any **FIVE** of the following problems. Each problem counts 20 points.

1. For the solution of the linear system $A\mathbf{x} = \mathbf{b}$ with $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, consider the following iterative method $\mathbf{x}^{(k+1)} = B(\theta)\mathbf{x}^{(k)} + \mathbf{g}(\theta)$, $k \geq 0$, with $\mathbf{x}^{(0)}$ given, where $\theta$ is a real parameter and
$$B(\theta) = \frac{1}{4}\begin{bmatrix} 2\theta^2 + 2\theta + 1 & -2\theta^2 + 2\theta + 1 \\ -2\theta^2 + 2\theta + 1 & 2\theta^2 + 2\theta + 1 \end{bmatrix}, \quad \mathbf{g}(\theta) = \begin{bmatrix} \frac{1}{4} - \theta \\ \frac{1}{4} - \theta \end{bmatrix}.$$
   (a) Check that the method is consistent $\forall\, \theta \in \mathbb{R}$.
   (b) Determine that values of $\theta$ for which the method is convergent.
   (c) Compute the optimal value of $\theta$ that maximizes the convergence rate.

2. Let $f \in C^2([a,b])$, $x_0, \ldots, x_n$ be $n+1$ distinct nodes of $[a,b]$ with $a = x_0 < x_1 < \cdots < x_n = b$, and the function $s(x)$ on the interval $[a,b]$ be a spline of degree 3 relative to the nodes.
   (a) Prove that $\displaystyle\int_a^b [s''(x)]^2\,dx \leq \int_a^b [f''(x)]^2\,dx$.
   (b) Verify the validity of part (a) in the case where the spline $s$ satisfies conditions of the form $s'(a) = f'(a)$, $s'(b) = f'(b)$.

3. Given $\displaystyle\int_{-1}^{1} |x|e^{x^3}\,dx$.
   (a) Apply the midpoint, trapezoidal, and Cavalieri-Simpson composite rules to approximate the integral.
   (b) Discuss their convergence in (a) as a function of the size $h$ of the subintervals.

4. Fix $0 < T < +\infty$. Given an initial-value problem (IVP): $y'(x) = f(x, y(x))$, $x \in I$ and $y(x_0) = y_0$, where $f(x,y)$ is a continuous real-valued function in $I \times (-\infty, +\infty)$ and $I = (x_0, x_0 + T)$. For $h > 0$, let $x_n = x_0 + nh$, with $n = 0, 1, 2, \ldots, N_h$, be the sequence of discretization nodes of $I$ into subintervals $I_n = [x_n, x_{n+1}]$, where $N_h$ is maximum integer such that $x_{N_h} \leq x_0 + T$. Let $u_j$ be the approximation at node $x_j$ of the exact solution $y(x_j)$ of the IVP. Consider the following family of linear multistep methods,
$$u_{n+1} = u_n + h\left[\left(1 - \frac{\alpha}{2}\right) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})\right],$$
depending on the real parameter $\alpha$.
   (a) Study their consistency as a function of $\alpha$.
   (b) Use the corresponding method in (a) to solve the Cauchy problem: $y'(x) = -10y(x)$ for $x > 0$ and $y(0) = 1$.
   (c) Determine the values of $h$ in correspondence of which the method is absolutely stable.

5. Given a two-point boundary value problem (BVP): $-u''(x) = f(x)$, $0 < x < 1$ and $u(0) = u(1) = 0$, where $f \in C^0([0,1])$. Let a function $G$ be defined in a piecewise linear function of $s$ for fixed $x$, $G(x,s) = s(1-x)$ if $0 \leq s \leq x$ and $G(x,s) = x(1-s)$ if $x \leq s \leq 1$. Consider that the approximation to the solution $u$ of the BVP is a finite sequence $\{u_j\}_{j=0}^n$ defined only at grid points $\{x_j\}_{j=0}^n$ (with the understanding that $u_j$ approximates $u(x_j)$) by requiring that $-\dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f(x_j)$, for $j = 1, \ldots, n-1$ and $u_0 = u_n = 0$, where $n \geq 2$, $h = 1/n$, and the grid points $\{x_j\}_{j=0}^n$ given by $x_j = jh$.
   Prove that $G^k(x_j) = hG(x_j, x_k)$, where $G^k$ is a grid function for the grid point $x_k$ and the solution to the following problem $L_h G^k = e^k$, where $e^k$ is the discretization error grid function of the BVP (it satisfies $e^k(x_j) = \delta_{kj}$, $1 \leq j \leq n-1$) and $L_h$ is the operator defined by
$$(L_h u)(x_j) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \quad j = 1, \ldots, n-1.$$

6. Given a partial differential equation (PDE) with the solution $u = u(x,t)$ for $x \in [0,1]$ and $t > 0$, $\frac{\partial u}{\partial t} + Lu = 0$, $0 < x < 1$, $t > 0$, subject to the boundary conditions $u(0,t) = u(1,t) = 0$, $t > 0$ and the initial condition $u(x,0) = u_0(x)$ for $0 \leq x \leq 1$, where $L$ is the differential operator defined as $Lu = -\nu \frac{\partial^2 u}{\partial x^2}$. First, for all $t > 0$, to multiply the PDE by a test function $v = v(x)$ and integrate over $(0,1)$. Then,

$$\int_0^1 \frac{\partial u(t)}{\partial t} v\,dx + a(u(t),v) = 0 \quad \forall\, v \in \mathrm{H}_0^1(0,1) \text{ with } u(0) = u_0,$$

where $a(u(t),v) = \int_0^1 \nu(\partial u(t)/\partial x)(\partial v/\partial x)\,dx$ is the bilinear form with the elliptic operator $L$. Next, consider the semi-discretization (to get a Galerkin formulation), and apply the implicit Euler scheme for the temporal variable to the Galerkin problem, and obtain

$$\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}, v_h\right) + a(u_h^{k+1}, v_h) = 0 \quad \forall\, v_h \in V_h,$$

where $V_h$ is a suitable finite dimensional subspace of $\mathrm{H}_0^1(0,1)$, $\Delta t > 0$, and $(u_h, v_h) = \int_0^1 u_h v_h\,dx$.

Prove that $\|u_h^{k+1}\|^2_{\mathrm{L}^2(0,1)} + 2\nu\Delta t \left\|\dfrac{\partial u_h^{k+1}}{\partial x}\right\|^2_{\mathrm{L}^2(0,1)} \leq \|u_h^k\|^2_{\mathrm{L}^2(0,1)}$.