(1) (10%) Determine the nature and stability properties of the critical points of the system, and sketch the phase diagram.

$$\begin{cases} x' = x^2 + y^2 - 4, \\ y' = y - 2x. \end{cases}$$

(2) (10%) If $0 < \epsilon \ll 1$, derive a two-term perturbation approximation to the problem

$$\begin{cases} y'' + 9y = 3\epsilon y^3, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

(3) (15%) If $0 < \epsilon \ll 1$, derive a uniform approximation to the problem

$$\begin{cases} \epsilon y'' - (2x+1)y' + 2y = 0 \quad \text{in } [0,1], \\ y(0) = 1, \quad y(1) = 0. \end{cases}$$

(4) (10%) If $0 < \epsilon \ll 1$, find the WKB approximation to the problem

$$\begin{cases} \epsilon y'' - (1+x^2)y = 0 \qquad \text{in } (0,\infty), \\ y(0) = 0, \quad \lim\limits_{x\to\infty} y(x) = 0. \end{cases}$$

(5) (10%) Determine the extremal of the functional

$$J(y) = \int_1^2 \frac{\sqrt{1+|y'|^2}}{x} dx$$

for $y \in C^1[1,2],\ y(1) = 0,\ y(2) = 1.$

(6) (10%) Find the solution of the problem

$$y' + 2y(t-\pi) = \sin t \qquad \text{for } t \in R.$$

(7) (10%) Find all of the eigenvalues and eigenfunctions to the problem

$$\begin{cases} -y'' = \lambda y \qquad \text{in } (0,\pi), \\ y(0) = 0, \qquad y(\pi) = 0. \end{cases}$$

(8) (10%) If $H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ and $f(x) = H(x) - H(-x)$ in $(-1,1)$, find the weak derivative of f.

(9) (15%) A bead of mass m with initial velocity 0 slides with no friction under the force of gravity g from a point $(0,b)$ to a point $(a,0)$ along a wire defined by a curve $y = y(x)$ in the xy plane. How to find a curve that leads to the fastest time of descent?