National Yang Ming Chiao Tung University
Department of Applied Mathematics

Real Analysis Ph.D. Qualifying Exam, Spring 2024

> There are 7 question sets of total 100 points.
> Answer questions as carefully and completely as possible.
> Do not make formal arguments without mathematical justification.
> If you use a major theorem, mention it by name and check its hypotheses.

Setting: In the following problems, whenever not specified, the sets are assumed be Lebesgue measurable subsets of some Euclidean spaces $\mathbb{R}^n$ and integrations are Lebesgue integrals. We write $m(E)$ to be the Lebesgue measure of a measurable set E in $\mathbb{R}^n$. The notation "a.e." stands for "almost everywhere".

1. (15 %) Construct a measurable subset E of $[0,1]$ such that for every subinterval I, both $E \cap I$ and $I \setminus E$ have positive measure.

2. Let $E \subset \mathbb{R}^n$ be a measurable set, and we say that x is a point of Lebesgue density of E if

$$\lim_{m(B)\to 0,\ x\in B} \frac{m(B\cap E)}{m(B)} = 1,$$

   where B is a ball center at x. Show that

   (a) (5 %) Almost every point $x \in E$ is a point of density of E.

   (b) (5 %) Almost every point $x \notin E$ is not a point of density of E.

3. (15 %) Show that the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

4. (10 %) Let f be a measurable function defined on E, and define $\omega_f(a) := m(x \in E : f(x) > a)$ for $-\infty < a < \infty$. If $f_k \nearrow f$[1], show that $\omega_{f_k} \nearrow \omega_f$.

5. (15 %) Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ as $k \to \infty$ and $f_k \le f$ a.e. on E. Show that $\int_E f_k \to \int_E f$ as $k \to \infty$.

6. (15 %) Let f be measurable and finite a.e. on $[0,1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$. Show that f is integrable on $[0,1]$.

7. (20 %) Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}.$$

[1] The notation is to denote $f_k \le f_{k+1}$ for all $k \in \mathbb{N}$, and $f_k \to f$ a.e. as $k \to \infty$.