NATIONAL YANG MING CHIAO TUNG UNIVERSITY
2024 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 112-2

1. (10 %) Find the stable, unstable and center subspaces E^s, E^u and E^c of $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} -1 & -3 & 0 \\ 0 & 2 & 0 \\ 5 & 7 & -1 \end{pmatrix}.$$

2. (15 %) Solve the nonhomogeneous ODE:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t),\ t > 1, \quad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{R}^2$ and

$$\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 2/t^2 & -2/t \end{pmatrix}, \quad \mathbf{B}(t) = \begin{pmatrix} 6t \\ 9/t^4 \end{pmatrix}.$$

3. (15 %) Consider the following prey-predator system

$$(\mathbf{P}) \begin{cases} \dot{x} = rx\big(1 - \dfrac{x}{K}\big) - \alpha xy, \\ \dot{y} = y(cx - d), \\ x(0) > 0, \quad y(0) > 0, \end{cases}$$

where $r, K, \alpha, c, d > 0$.

(a) (7 %) Prove that the solution of **(P)** exists and is unique for $t > 0$.

(b) (8 %) Prove that the solution of **(P)** is positive and bounded for all $t > 0$.

4. (25 %) Consider the following system

$$(\mathbf{Q}) \begin{cases} \dot{x} = x - y - 2x^3 - 3xy^2, \\ \dot{y} = x + y - 2x^2y - 3y^3. \end{cases}$$

(a) (5 %) State the Poincaré-Bendixon Theorem for autonomous systems $\dot{\mathbf{x}} = f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$.

(b) (5 %) State Dulac's Criteria for autonomous systems $\dot{\mathbf{x}} = f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$.

(c) (10 %) Prove that there exists a periodic solution for **(Q)** in an annulus.

(d) (5 %) Prove or disprove the uniqueness of the periodic solution for **(Q)**.

5. (20 %) Consider the second-order differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

for some f and g are smooth real-valued functions defined in a neighborhood of the origin.

(a) (5 %) Rewrite the equation as

$$\begin{cases} \dot{u} = v - F(u), \\ \dot{v} = -g(u) \end{cases}$$

for some F.

(b) (5 %) State the definition of Lyapunov function for autonomous systems $\dot{\mathbf{x}} = f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^2$.

(c) (10 %) Let $G(u) := \int_0^u g(s)ds$. Assume that $G(u) > 0$ and $g(u)F(u) > 0$ in a deleted neighborhood of the origin. Prove or disprove that the origin is asymptotically stable.

6. (15 %) Consider the linear system $\dot{\mathbf{x}} = A(t)\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$ is a continuous, periodic matrix with periodic $T > 0$. Let $\Phi(t)$ be a fundamental matrix of $\dot{\mathbf{x}} = A(t)\mathbf{x}$.

(a) (5 %) Prove that $\Phi(t+T)$ is also a fundamental matrix of $\dot{\mathbf{x}} = A(t)\mathbf{x}$.

(b) (10 %) Prove that there exists a periodic nonsingular matrix $P(t)$ of period T and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.