NATIONAL YANG MING CHIAO TUNG UNIVERSITY

2023 Real Analysis Ph.D. Qualifying Exam

Academic Year 112-1, 2023

INSTRUCTIONS: In the following problems, whenever not specified, the sets are assumed be Lebesgue measurable subsets of some Euclidean spaces $\mathbb{R}^n$ and integrations are Lebesgue integrals. You may use any standard theorem from your real analysis course. You can either mention the theorem by name or provide its complete statement. However, ensure that you verify the fulfillment of the theorem's assumptions before applying it.

1. (15 %) Give an example of a sequence of continuous functions f_n on $\mathbb{R}$ such that $f_n \to 0$ pointwise everywhere, but, for every continuous function g,
$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)g(x)dx = g(0).$$

2. (15 %) Show that the closed unit ball in ℓ^2 is not compact.

3. (15 %) Show that $C^1[a,b]$ with norm
$$|f|_{C^1} = \sup_{x\in[a,b]}|f(x)| + \sup_{x\in[a,b]}|f'(x)|$$
is complete.

4. (15 %) Is the following statement true?
For $f \in L^1(\mathbb{R})$,
$$\lim_{n\to\infty}\int_{|x|<\frac{1}{n}}|f(x)|dx = 0$$
If this statement is true, please provide a proof; if it is not true, please provide a counterexample

5. (20 %) Let A and B be bounded Lebesgue measurable subsets of $\mathbb{R}$, both with positive measure. Let χ_A and χ_B be the characteristic functions of A and B.
(a) Show that the convolution $\chi_A \star \chi_B$ is a continuous function and $\int_{\mathbb{R}} \chi_A \star \chi_B > 0$.
(b) Show that $A + B := \{x + y : x \in A, y \in B\}$ contains a nonempty open interval.

6. (20 %) Consider $\mathbb{R}$, equipped with the Lebesgue measure. Suppose that $f \in L^\infty(\mathbb{R})$ is such that, for every $g \in L^1(\mathbb{R})$ and every $a \in \mathbb{R}$, we have
$$\int_{-\infty}^{\infty} g(x)[f(x+a) - f(x)]dx = 0.$$
Prove that there exists a constant c such that $f(x) = c$ for almost all $x \in \mathbb{R}$.