NYCU Qualifying Exam: Partial Differential Equations, September 13, 2023

Instruction : Please answer all of the following problems. Each answer which you give should be supported by rigorous mathematical arguments.**Time allowed: 4 hours**

Problem 1. Let $\lambda > 0$ be a prescribed constant. Consider the linear operator

$$\mathcal{L} : C^2\big(\Omega\big) \to C^0\big(\Omega\big) \tag{0.1}$$

which is given by

$$\mathcal{L}u = \triangle u + \lambda u, \tag{0.2}$$

in which Ω can be any domain in $\mathbb{R}^3$.

Part (a)(10 points) Let $O = (0,0,0)$ be the origin in the Euclidean space $\mathbb{R}^3$. Consider $w \in C^2\big(\mathbb{R}^3 - \{O\}\big)$ be a solution to the equation $\mathcal{L}w = 0$ on $\mathbb{R}^3 - \{O\}$ which is radially symmetric in that w can be written in the form of $w(x) = F(|x|)$, for all $x \in \mathbb{R}^3 - \{O\}$. Find the most general expression of w, and give your answer in terms of functions of $|x|$. (**Hint :** You can use the fact that a solution ψ to the O.D.E. $\psi''(r) + \lambda\psi(r) = 0$ can always be written as a linear combination of $\cos(\sqrt{\lambda}r)$ and $\sin(\sqrt{\lambda}r)$).

Part (b)(10 points) Find a function $K \in C^2\big(\mathbb{R}^3 - \{O\}\big) \cap L^1_{loc}\big(\mathbb{R}^3\big)$ which satisfies the following relation for all test functions $\phi \in C_c^\infty\big(\mathbb{R}^3\big)$

$$\int_{\mathbb{R}^3} K(x)\triangle\phi(x)\mathrm{d}x + \lambda \int_{\mathbb{R}^3} K(x)\phi(x)\mathrm{d}x = \phi(0). \tag{0.3}$$

(**Hint :** Such a function K can be chosen to be radially symmetric, and you may rely on your work in **Part (a)**).

Problem 2. Take $N \geq 1$ be an integer. Let $f \in C^0\big(\mathbb{R}^N\big) \cap L^\infty\big(\mathbb{R}^N\big)$. Consider the function $u : (0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ which is given by

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} f(y)\mathrm{d}y, \tag{0.4}$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^N$. It is well-known that the function u as given by (0.4) is a solution to the heat equation $\partial_t u - \triangle u = 0$ on $(0,\infty) \times \mathbb{R}^N$.

Part (a) (12 points) In this problem, prove that the following relation holds for each $x_0 \in \mathbb{R}^N$.

$$\lim_{(t,x)\to(0,x_0)} u(t,x) = f(x_0)$$

Part (b) (12 points) Here, we specialize to the case of $N = 1$, and suppose that f satisfies the stronger condition $f \in L^2\big(\mathbb{R}\big) \cap C^0\big(\mathbb{R}\big) \cap L^\infty\big(\mathbb{R}\big)$, prove that there exists some absolute constant $C_0 > 0$, such that the function u as given by (0.4) satisfies the following *gradient-estimate*

$$\int_{\mathbb{R}} \Big|\frac{\partial u}{\partial x}(t,x)\Big|^2 \mathrm{d}x \leq \frac{C_0}{t}\|f\|^2_{L^2(\mathbb{R})}, \tag{0.5}$$

for all $t > 0$.

Problem 3.(**14 points**) Here, we consider the closed cube $[-1,1]\times[-1,1]=\{(x,y): |x|\le 1, |y|\le 1\}$ in $\mathbb{R}^2$. Consider a function $u\in C^2\big([-1,1]\times[-1,1]\big)$ which satisfies the following conditions.

- $\triangle u = 0$ holds on $(-1,1)\times(-1,1)=\{(x,y): |x|<1, |y|<1\}$.
- $u(1,y)=u(-1,y)=0$ holds for all $y\in[-1,1]$.
- $\frac{\partial u}{\partial x}(x,1)=\frac{\partial u}{\partial y}(x,1)$ and $\frac{\partial u}{\partial x}(x,-1)=\frac{\partial u}{\partial y}(x,-1)$ holds for all $x\in[-1,1]$.

Prove that $u=0$ holds on $[-1,1]\times[-1,1]$.
(**Hint :** You may start with the identity $0=\int_{[-1,1]\times[-1,1]} u(x,y)\triangle u(x,y)\mathrm{d}x\mathrm{d}y$, and try to perform integration by parts by using the prescribed boundary conditions)

Problem 4. Part (**a**) (**10 points**) Take an integer $N\ge 2$. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider a function $u\in C^1\big(\overline{\Omega}\big)\cap C^\infty\big(\Omega\big)$ which is harmonic on Ω. Prove that the following relation holds.

$$\max_{\overline{\Omega}}\big|\nabla u\big| = \max_{\partial\Omega}\big|\nabla u\big|. \tag{0.6}$$

Part (**b**)(**12 points**) In particular, take $N=2$, and we consider the open ball $B_0(1)=\{(x,y)\in\mathbb{R}^2: x^2+y^2<1\}$ in $\mathbb{R}^2$. Consider a function $u\in C^1\big(\overline{B_0(1)}\big)\cap C^\infty\big(B_0(1)\big)$ which is harmonic on $B_0(1)$. Suppose that there are two functions $\Psi_1,\Psi_2\in C^1\big(\overline{B_0(1)}\big)\cap C^2\big(B_0(1)\big)$ satisfying the following properties.

- $\triangle\Psi_1\le 0$ and $\triangle\Psi_2\ge 0$ hold on $B_0(1)$. Moreover, $\Psi_1=u=\Psi_2$ holds on $\partial B_0(1)$.

Prove that the following relation holds:

$$\max_{\overline{B_0(1)}}\big|\nabla u\big| \le \max\Big\{\max_{\partial B_0(1)}\big|\nabla\Psi_1\big|, \max_{\partial B_0(1)}\big|\nabla\Psi_2\big|\Big\}.$$

Problem 5 (**10 points**) Take an integer $N\ge 2$. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For a prescribed $T>0$, define the space-time region $\Omega_T=(0,T]\times\Omega$, and also $\Gamma_T=\overline{\Omega_T}-\Omega_T$. (note that $\overline{\Omega_T}=[0,T]\times\overline{\Omega}$). Consider a function $u\in C^2\big(\overline{\Omega_T}\big)$ which satisfies the following properties.

- $\partial_t^2 u-\triangle u=0$ holds on Ω_T. That is, u solves the wave equation on Ω_T.
- $u(t,x)=0$ holds for all $(t,x)\in\Gamma_T$.
- $\partial_t u(0,x)=0$ holds for all $x\in\Omega$.

Use energy method or otherwise to prove that $u(t,x)=0$ holds for all $(t,x)\in\Omega_T$.

Problem 6 (**10 points**) Consider the piece-wise linear, continuous function $\Phi:\mathbb{R}\to[0,\infty)$ which is given by $\Phi(x)=x\chi_{\{x>0\}}$ for all $x\in\mathbb{R}$. (Note that equivalently, we have $\Phi(x)=0$, for all $x\le 0$. While $\Phi(x)=x$ for all $x>0$). Consider now the region

$$\Omega=\Big\{(x,y)\in\mathbb{R}^2: y>\Phi(x)\Big\} \tag{0.7}$$

Let $g\in C^\infty\big((0,\infty)\big)$ be prescribed. Use the method of characteristics or otherwise to find an explicit expression for the solution $u\in C^1\big(\overline{\Omega}-\{(0,0)\}\big)$ to the equation

$$x\frac{\partial u}{\partial y}(x,y)-y\frac{\partial u}{\partial x}(x,y)=u(x,y)$$

on Ω, which satisfies the prescribed boundary condition $u(x,x)=g(x)$ for all $x>0$.