NATIONAL YANG MING CHIAO TUNG UNIVERSITY
2023 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 112-1, 2023

1. (10 %) Define

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}.$$

Solve the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

2. (20 %) Let f be real-valued and continuous on $D := \{(t, x) | 0 \le t \le a, -\infty < x < \infty\}$ for some $a > 0$. Suppose that $\hat{x}$ is a solution of $\dot{x} = f(t, x)$ with $\hat{x}(0) = x_0 \in \mathbb{R}$ defined for $0 \le t < \tau$ for some $\tau \in (0, a)$.

   (a) (10 %) Prove that either $\hat{x}$ can be extended as a solution beyond τ, or $|\hat{x}(t)| \to \infty$ as $t \to \tau^-$.

   (b) (10 %) Suppose that there exists a continuous function $\phi(r)$ defined in $0 \le r < \infty$ such that $|f(t, x)| < \phi(|x|)$ in D and for some $K \in [0, \infty)$, $\int_K^\infty \frac{dr}{\phi(r)} = \infty$. Prove or disprove that the solution $\hat{x}$ exists on $[0, a]$.

3. (20 %) Consider the following system

$$\begin{cases} \dot{x} = 5x(1 - x - 2y), \\ \dot{y} = 2y(1 - y - 3x), \end{cases}$$

where $x(t)$ and $y(t)$ represent the population size of two interacting species at time t, respectively.

   (a) (5 %) Find all equilibria with nonnegative components.

   (b) (10 %) Do the local stability analysis (asymptotically stable/stable/unstable) at each equilibrium and classify them as center, spiral, node or saddle. Also, please explain why Hartman–Grobman theorem is needed.

   (c) (5 %) Predict the global asymptotic behavior of $(x(t), y(t))$ when $x(0) > 0$ and $y(0) > 0$. Please write down your reasons as thoroughly as possible.

4. (18 %) Consider the system

$$\begin{cases} \dot{x} = 2x + 2y - x(2x^2 + y^2), \\ \dot{y} = -2x + 2y - y(2x^2 + y^2). \end{cases}$$

(a) (6 %) Rewrite the system in polar coordinates (r, θ).

(b) (12 %) Prove or disprove that the system admits a limit cycle on the phase plane.

5. (12 %) Let $n \times n$ matrix $A(t)$ and $n \times 1$ vector $b(t)$ be continuous on $\mathbb{R}$, and periodic of period $T > 0$. That is,

$$A(t+T) = A(t), \quad b(t+T) = b(t) \quad \forall t.$$

Prove that if the differential system $\dot{\mathbf{x}} = A(t)\mathbf{x}$ has no periodic solution of period T other than the trivial solution $\mathbf{x} \equiv \mathbf{0}$, then the differential system $\dot{\mathbf{x}} = A(t)\mathbf{x} + b(t)$ has a unique periodic solution of period T.

6. (20 %) Consider the system

$$\begin{cases} \dot{x} = x^2 + y - y^2, \\ \dot{y} = -x - 2xy. \end{cases}$$

(a) (5 %) Prove that the system is a Hamiltonian system and find the Hamiltonian $H(x, y)$.

(b) (5 %) Prove that the trajectories lie on the contours defined by $H(x, y) = C$, where C is a constant.

(c) (5 %) Prove or disprove that $(0, 0)$ is asymptotically stable.

(d) (5 %) Sketch the phase portraits including the stable and unstable manifolds at all equilibria.