NATIONAL YANG MING CHIAO TUNG UNIVERSITY

Real Analysis Ph.D. Qualifying Exam, Spring 2023

1. Let $1 \le p < \infty$. All parts refer to Lebesgue measure on $\mathbb{R}$.
   (a) (5%) Give an example where $\{f_n\}$ converges to f pointwise, $\|f_n\|_p < M < \infty$ for all n, and $\|f_n - f\|_p \not\to 0$.
   (b) (10%) Show that if $\{f_n\}$ converges to f pointwise and $\|f_n\|_p \to \|f\|_p$, then $\|f_n - f\|_p \to 0$.

2. (a) (10%) Given $n \in \mathbb{N}$, show that $(1 + \frac{x}{n})^n \le e^x$ for $x \ge 0$.
   (b) (5%) Evaluate
$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx.$$

3. (a) (10%) Suppose that $0 < p < q < r \le \infty$. Show that if $f \in L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, then
$$\|f\|_q \le \|f\|_p^\lambda \|f\|_r^{1-\lambda},$$
   where $\lambda \in (0,1)$ is defined by $q^{-1} = \lambda p^{-1} + (1-\lambda) r^{-1}$. (Hint: Consider two cases: (i) $r = \infty$, (ii) $r < \infty$.)
   (b) (10%) Assume $f \in L^r(\mathbb{R}^n)$ for some $0 < r < \infty$. Show that
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

4. Suppose that $\phi \in L^1(\mathbb{R}^n)$ is nonnegative with integral 1. Set $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$, $\epsilon > 0$.
   (a) (10%) Prove that for all $M > 0$,
$$\lim_{\epsilon\to 0} \int_{\{\|x\| > M\}} \phi_\epsilon(x) dx = 0.$$
   (b) (10%) Let $f \in L^\infty(\mathbb{R}^n)$. Prove that
$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y) dy = f(x)$$
   at every point x of continuity of f.

5. (10%) Suppose that $x_1, \dots, x_n$ are linearly independent elements of a normed linear space X. Show that there is a constant $c > 0$ such that
$$\|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \ge c(|\lambda_1| + \cdots + |\lambda_n|)$$
   for all scalars $\lambda_1, \dots, \lambda_n$.

6. (20%) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of absolutely continuous functions such that $f_n(0) = 0$ for all n. Suppose that $\{f_n'\}$ is a Cauchy sequence in $L^1([0,1])$. Show that there is an absolutely continuous function f such that $f_n \to f$ uniformly on $[0,1]$.