NATIONAL YANG MING CHIAO TUNG UNIVERSITY

2023 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 111-2, 2023

1. (15 %) Prove the following Grownwall's inequality:
   Suppose that $g(t)$ is a continuous real valued function that satisfies $g(t) > 0$ and

$$g(t) \leq C + K \int_0^t g(s)ds,$$

   for $t \in [0, a]$, where C, K, a are positive constant. Show that for $t \in [0, a]$,

$$g(t) \leq Ce^{Kt}.$$

2. (15 %) Consider the linear inhomogeneous ODE for $y(t)$,

$$y'' + 4y = f(t)$$

   where $f(t)$ is a certain given function. Use the method of variation of parameters to obtain the general solution to this equation.

3. (15 %) Consider the nonlinear system

$$\begin{cases} x_1' = -x_1 \\ x_2' = -x_2 + x_1^2 \\ x_3' = x_3 + x_1^2 \end{cases}$$

   (a) (5 %) Find the equilibrium

   (b) (10 %) Find the stable and unstable subspace at equilibrium

4. (15 %) Does

$$\begin{cases} x' = x(1-x) - 2xy \\ y' = 2y(1-y) - xy \end{cases}$$

   have periodic solution in the first quadrant?(Please state which theorem you use)

5. (15 %) Consider the following second order ODE

$$y'' + (1 + R(t))\, y = 0. \tag{1}$$

   Assume that $\int_0^\infty |R(t)|dt < +\infty$. Show that every solutions of (1) is bounded as $t \to \infty$.

6. (25 %) Consider the following system of ODEs:

$$\begin{cases} x' &= (1+x)\sin y \\ y' &= 1 - x - \cos y \end{cases}$$

(a) (5 %) Determine all the equilibrium.

(b) (5 %) Determine the corresponding linear system near each equilibrium.

(c) (8 %) Find the eigenvalues of each linear system and determine the stability of the linear system.

(d) (7 %) Draw a phase portrait of the nonlinear system to show whether the linear system provides the definite information about the nonlinear system near equilibrium.