PhD Qualifying Exam in Numerical Analysis

Spring 2023

1. Consider the iteration method
$$Ax^{(k+1)} = Bx^{(k)} + f,$$
where $x^{(k)}, f$ are vectors in $\mathbb{R}^n$ and A, B are square matrices in $\mathbb{R}^{n\times n}$. Assume that A is nonsingular and $q = \|A^{-1}\|\|B\| < 1$ where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm in $\mathbb{R}^n$. Please answer the following questions:
   (a) (7 pts) Show that the linear system $Ax = Bx + f$ has a unique solution.
   (b) (7 pts) Show that the iterative solution $x^{(k)}$ converges to the solution x of the linear system in part (a) for any initial guess $x^{(0)}$.
   (c) (7 pts) Continued in part (b), derive a error estimate $\|x - x^{(k)}\|_2$ in terms of k, q and $\|x^{(1)} - x^{(0)}\|_2$.

2. Given a function $f(x) = \frac{1}{x^2+1}$ defined on the interval $[-5, 5]$. Let $f_n(x)$ be a function that interpolates $f(x)$ by using Lagrange interpolation on equally spaced nodes $\Delta x = \frac{10}{n}$ where $n+1$ is the number of nodes. Please answer the following questions:
   (a) (7 pts) Show that for any $x \in [-5, 5]$, we have
   $$f(x) - f_n(x) = w_n(x) f[x_0^{(n)}, \cdots, x_n^{(n)}, x]$$
   where $x_j^{(n)} = -5 + j\Delta x, j = 0, \cdots, n$, $w_n(x) = \Pi_{j=0}^{n}(x - x_j^{(n)})$ and $f[x_0^{(n)}, \cdots, x_n^{(n)}, x]$ is the $(n+1)$st Newton divided difference of f.
   (b) (7 pts) Show that
   $$f[x_0^{(n)}, \cdots, x_n^{(n)}, x] = \frac{1}{iw_n(i)} \frac{r_n}{x^2+1},$$
   where i is the imaginary number and $r_n = x$ if n is even; $r_n = i$ if n is odd.
   (c) (7 pts) Show that f_n does not converges to f at $x = 4$ as n goes to infinity.

3. Given $n+1$ distinct points $a = x_0 < x_1 < \cdots < x_n = b$ on the interval $[a, b] (a < b)$ and $f \in C^2([a, b])$, please answer the following questions:
   (a) (7 pts) Show that there exists a unique piecewise-polynomial $S(x)$ with degree 3 such that $S(x) = S_j(x)$ for $x \in [x_j, x_{j+1}], j = 0, \cdots, n-1$, satisfying
   (1) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for $j = 0, \cdots, n-1$;
   (2) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for $j = 0, \cdots, n-2$;
   (3) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for $j = 0, \cdots, n-2$;
   (4) $S''(x_0) = S''(x_n) = 0$.

(b) (7 pts) Show that $S(x)$ in part (a) satisfies

$$\int_a^b (S''(x))^2\,dx \le \int_a^b (f''(x))^2\,dx$$

where the equality holds if and only if $f(x) = S(x)$ on $[a, b]$.

4. (9 pts) Given the ordinary differential equation $\frac{dy}{dt} = f(t, y(t))$ with the initial condition $y(0) = y_0$. Consider the family of linear multistep methods

$$y_{n+1} = \alpha y_n + \frac{h}{2}(2(1-\alpha)f_{n+1} + 3\alpha f_n - \alpha f_{n-1}),$$

where h is the time step and α is a real parameter. Analyze consistency and order of the methods as functions of α, determining the values α^* for which the resulting method has maximal order.

5. Consider the two-point boundary value problem

$$(P1)\begin{cases} -u''(x) + u(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

We can rewrite problem $(P1)$ as the following weak formulation:

$$(P2): \text{ Find } u \in V \text{ satisfying } a(u,v) = (f,v), \text{ for all } v \in V,$$

where the functional space $V = H_0^1(0,1)$, $(f,v) = \int_0^1 fv\,dx$ denotes the scalar product of $L^2(0,1)$ and the bilinear form $a(u,v) : V \times V \to \mathbb{R}$ is defined by

$$a(u,v) = \int_0^1 u'v'\,dx + \int_0^1 uv\,dx.$$

We use the Galerkin method to solve $(P2)$ by introducing V_h as a finite dimensional vector subspace of V and approximate $(P2)$ by the problem

$$(P3): \text{ Find } u_h \in V_h \text{ satisfying } a(u_h, v_h) = (f, v_h), \text{ for all } v_h \in V_h,$$

Please answer the following questions:

(a) (7 pts) If $f \in C^0([0,1])$, show that $\|u\|_\infty \le \frac{1}{2}\frac{e}{e-e^{-1}}\|f\|_\infty$.

(b) (7 pts) If u and u_h are the solutions of $(P2)$ and $(P3)$ respectively and $f \in L^2(0,1)$, show that

$$|u - u_h|_{H^1(0,1)} \le c \min_{w_h \in V_h} |u - w_h|_{H^1(0,1)},$$

where c is a positive constant independent of h and we endow the space $H_0^1(0,1)$ with the following norm

$$|v|_{H^1(0,1)} = \left(\int_0^1 (v'(x))^2\,dx\right)^{1/2}.$$

6. Assume that $u = u(x,t)$ satisfies the heat equation

$$(P4): u_t - \nu u_{xx} = f, \quad (x,t) \in (0,1) \times (0,\infty),$$

subject to the boundary condition

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

and the initial value condition

$$u(x,0) = u_0(x), x \in [0,1],$$

where $\nu > 0$ is the diffusive constant and $f(x,t)$ is the forcing term. For the solution $u(x,t)$ of $(P4)$, we define the energy $E(t)$ on the time interval as

$$E(t) = \int_0^1 u^2(x,t)\, dx.$$

Now, consider the central difference for the spatial variable and θ method for the temporal variable and obtain the following numerical scheme

$$(P5): \begin{cases} \frac{u_i^{k+1} - u_i^k}{\Delta t} - \nu\theta \frac{u_{i+1}^{k+1} - 2u_i^{k+1} + u_{i-1}^{k+1}}{(\Delta x)^2} - \nu(1-\theta)\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} \\ = \theta f_i^{k+1} + (1-\theta) f_i^k, i = 1, \cdots, n-1; k = 0, 1, \cdots; \\ u_0^k = u_n^k = 0, k = 0, 1, \cdots; \\ u_i^0 = u_0(x_i), i = 1, \cdots, n-1, \end{cases}$$

where $\Delta x = \frac{1}{n}, \Delta t > 0$ and $\theta > 0$. Here, u_i^k is the approximation of u at $(x,t) = (i\Delta x, k\Delta t)$ and $f_i^k = f(i\Delta x, k\Delta t)$. We define $U^k = (u_1^k \quad u_2^k \quad \cdots \quad u_{n-1}^k)^T$ and rewrite $(P5)$ in terms of the matrix form as follows:

$$(P6): U^{k+1} - \theta\Delta t A U^{k+1} = U^k + (1-\theta)\Delta t A U^k + \theta \Delta t F^{k+1} + (1-\theta)\Delta t F^k,$$

where A is some $(n-1) \times (n-1)$ matrix and F^k is related to the forcing term f^k. Please answer the following questions:

(a) (7 pts) Show that $E(t)$ satisfies the following inequality for $t \geq 0$,

$$E(t) \leq e^{-\gamma t}(E(0)) + \frac{1}{\gamma}\int_0^t e^{\gamma(s-t)} F(s)\, ds,$$

where $\gamma = \frac{\nu}{c_p^2}$, $F(t) = \int_0^1 f^2(x,t)\, dx$ and c_p is the Poincare constant.

(b) (7 pts) Show that the problem $(P6)$ is solvable for any given U^k, $\Delta t > 0$ and $\theta > 0$.

(c) (7 pts) Find all values of θ to determine the numerical scheme $(P5)$ is unconditionally stable.