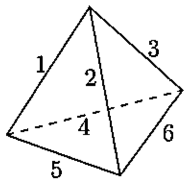
NYCU DEPARTMENT OF APPLIED MATHEMATICS
QUALIFYING EXAMINATION IN ALGEBRA
FOR THE PH. D. PROGRAM

Show all your work and carefully justify all your answers. Answers without explanation will not receive any score.

1. Let G be a group of order 255.
   a)**(5 pts)** Show that G contains a normal subgroup of order 17.
   b)**(5 pts)** Show that G contains a normal cyclic subgroup of order 85.
   c)**(10 pts)** Find the all possible structures of G.

2. Consider a tetrahedron frame assembled using 6 bonds in n colors and let G be its group of rotational symmetries. Regard G as a subgroup of S_6 using the following labelling on 6 edges.



   a)**(5 pts)** Calculate the size of G.
   b)**(5 pts)** List the cycle structures of elements in G and their corresponding number of elements.
   c)**(10 pts)** Calculate the number of ways to assemble the tetrahedron frame using bonds in n colors, taking into account the action of G.
   d)**(5 pts)** Calculate the number of ways to assemble the tetrahedron frame using 2 red bonds, 2 blue bonds, and 2 yellow bonds, taking into account the action of G.

3. Let $E = \mathbb{Q}(\sqrt[3]{2}, \omega)$ be a finite extension over $\mathbb{Q}$, where $\omega = \exp(2\pi i/3)$.
   a) **(5 pts)** Show that $E/\mathbb{Q}$ is a Galois extension.
   b) **(5 pts)** Calculate the degree of $[E : \mathbb{Q}]$.
   c) **(5 pts)** Find a basis of E as a vector space over $\mathbb{Q}$.
   d) **(5 pts)** Find the group structure of $\mathrm{Gal}(E/\mathbb{Q})$.
   e) **(10 pts)** List all subextensions of $\mathbb{Q}$ in E.

4. Let $R_p = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}$, where p is a odd prime.
   a) **(5 pts)** Show that R_p is a commutative subring of $M_2(\mathbb{Z}_p)$.
   b) **(10 pts)** For $p = 5$, list all elements of the group of units $R_5^\times$ and determine its structure as a product of cyclic groups.
   c) **(10 pts)** Show that R_p is an integral domain if and only if $p \equiv 3 \mod 4$.

Date: 2023 Spring.