National Yang Ming Chiao Tung University
Department of Applied Mathematics

Real Analysis Ph.D. Qualifying Exam, Fall 2022

There are 8 question sets of total 100 points.
Answer questions as carefully and completely as possible.
Do not make formal arguments without mathematical justification.
If you use a major theorem, mention it by name and check its hypotheses.

1. Let $\{f_n\}_n$ be a sequence of functions in $L^5([0,1])$ with respect to the Lebesgue measure. Determine if the following statements are true or false and justify your answers.
   (a) (5 %) If f_n converges pointwise almost everywhere to f, then there is a subsequence of $\{f_n\}_n$ converges f in L^5.
   (b) (5 %) If f_n converges in measure to f, then $\{f_n\}_n$ converges f in L^5.
   (c) (5 %) If f_n converges to f in L^5, then f_n converges to f in measure.
   (d) (5 %) If f_n converges to f in L^5, then there is a subsequence of $\{f_n\}_n$ converges pointwise almost everywhere to f.

2. (10 %) Let $\mathcal{F} = \left\{ f : [0,\infty) \longrightarrow [0,\infty] \mid \int_{[0,\infty)} f^5(x)\,dx \le 1 \right\}$. Compute $\sup_{f\in\mathcal{F}} \int_{[0,\infty)} f^4(x)e^{-x}\,dx$ explicitly.

3. Suppose $f : [0,1] \longrightarrow \mathbb{R}$ is a measurable function with respect to the Lebesgue measure on $[0,1]$ satisfying
$$\sum_{n=1}^{\infty} \left(\int_{[0,1]} |f(x)|^n\,dx \right) < \infty.$$
   (a) (5 %) Show that $|f(x)| < 1$ almost everywhere with respect to the Lebesgue measure on $[0,1]$.
   (b) (5 %) Show that the function $g(x) = \dfrac{1}{1-f(x)}$ is Lebesgue integrable with respect to the Lebesgue measure over $[0,1]$.

4. (10 %) If $f \in L^\infty(\mathbb{R})$ is a periodic function with period 1 and if
$$\int_{\mathbb{R}} f(x/\epsilon)\zeta(x)dx \longrightarrow \int_{\mathbb{R}} g(x)\zeta(x)dx \qquad \text{as } \epsilon \to 0,$$
for any $\zeta \in \mathscr{C}_0^1(\mathbb{R})$ = the set of functions with continuous derivatives and compact support, what is the function g? Why?

5. (10 %) Let B_r be a disk in $\mathbb{R}^2$ with radius r and centered at the origin, $f \in L^2(B_2)$, and

$$K = \begin{cases} 1 & \text{in } B_1, \\ 2 & \text{in } B_2 \setminus B_1. \end{cases}$$

If $U, \nabla U \in L^2(B_2)$ satisfies

$$\int_{B_2} K \nabla U \nabla \zeta \, dx = \int_{B_2} f \zeta \, dx \qquad \text{for any } \zeta \in \mathscr{C}_0^1(B_2),$$

in which region the Hessian $\nabla^2 U \in L^2$? Why?

6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $f(x, y) = |x| + |y|$ for any $(x, y) \in \mathbb{R}^2$. Define a Borel measure μ on $\mathbb{R}$ by $\mu(B) = \mathscr{L}^2(f^{-1}(B))$ for any Borel set $B \subset \mathbb{R}$, where $\mathscr{L}^2$ is the Lebesgue measure on $\mathbb{R}^2$.
   (a) (5 %) Show that μ is *absolutely continuous* with respect to the Lebesgue measure $\mathscr{L}^1$ on $\mathbb{R}$.
   (b) (5 %) Find the Radon-Nikodym derivative $\dfrac{d\mu}{d\mathscr{L}^1}$.

7. (15 %) If $(X, \mathfrak{B}, \mu)$ is a measurable space and $f \in L^p(X, \mu)$ for some $1 \leq p < \infty$, is $\lim_{p \to \infty} \|f\|_{L^p(X,\mu)} = \|f\|_{L^\infty(X,\mu)}$ true? Why?

8. Let $(X, \mathfrak{B}, \mu)$ be a measurable space and $1 \leq p < q \leq \infty$.
   (a) (10 %) Under what conditions $L^q(X, \mu) \subset L^p(X, \mu)$ is true? Why?
   (b) (5 %) Under what conditions $L^p(X, \mu) \subset L^q(X, \mu)$ is true? Why?