NATIONAL YANG MING CHIAO TUNG UNIVERSITY

2022 Ordinary Differential Equations Ph.D. Qualifying Exam

Academic Year 111-1, September 14, 2022

1. Consider a linear system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

   (a) (6 %) Compute the matrix exponential e^{At}.

   (b) (7 %) Find all $\mathbf{x}_0$ such that the solutions $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ are unbounded as $t \to \infty$.

   (c) (7 %) Find all $\mathbf{x}_0$ such that the solutions $\mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ are unbounded as $t \to -\infty$.

2. (10 %) Consider the following equation for $x(t)$:

$$\frac{dx}{dt} = x^\alpha$$

   with $x(0) \geq 0$ and $\alpha > 0$. Show that the only value of α such that the equation has solutions that are both unique and exist for all time is $\alpha = 1$.

3. Consider the second order differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + (x^2 - A^2) = 0$$

   (a) (7 %) Find the equilibrium point(s) and classify their types(sources, sinks, saddles, etc.) for $|A| > 1/8$.

   (b) (7 %) Find the equilibrium point(s) and classify their types(sources, sinks, saddles, etc.) for $|A| < 1/8$.

   (c) (6 %) Draw the trajectories in the phase plane for $|A| > 1/8$. (Please include trajectories that connect equilibrium points if any.)

4. Consider the system

$$\begin{cases} \dfrac{dx}{dt} = -y + xf(x,y), \\ \dfrac{dy}{dt} = x + yf(x,y), \quad (x,y) \in \mathbb{R}^2, \end{cases}$$

   where $f(x,y) = (x^2+y^2)^2 - 3(x^2+y^2) + 1$.