PhD Qualifying Exam in Numerical Analysis

Fall, 2022

(Total of 125 points)

1. A real-valued system of n linear equations in n unknowns consists of a set of algebraic relations and the system can be written in matrix form as $L\mathbf{x} = \mathbf{b}$. Let L be symmetric and positive definite with an additive splitting of the form $L = P - N$, where P and N are two suitable matrices. Suppose that the matrix $P + P^{\top} - L$ is positive definite.

   (a) (10 pts) Prove that P is invertible.

   (b) (10 pts) Given an iterative method defined in $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + P^{-1}\mathbf{r}^{(k)}$, where $\mathbf{r}^{(k)} = \mathbf{b} - L\mathbf{x}^{(k)}$. Prove that the iterative method is monotonically convergent with respect to norm $\|\cdot\|_L$.

   (c) (10 pts) Prove that $\rho(B) \le \|B\|_L < 1$, where B is the iteration matrix of the iterative method.

2. Let $g \in C^{n+1}[a,b]$ be a given real-valued function. Suppose that $x_0, x_1, \ldots, x_n$ are $n+1$ distinct real numbers in $[a,b]$.

   (a) (10 pts) Prove that there exists a unique polynomial Π_n of degree at most n such that $\Pi_n(x_i) = g(x_i)$ for $i = 0, 1, \ldots, n$.

   (b) (10 pts) Prove that for each x in $[a,b]$ there exists $\xi_x \in (a,b)$ such that

$$g(x) - \Pi_n(x) = \frac{1}{(n+1)!}\, g^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i).$$

3. (15 pts) Determine α and β so that the following quadrature rule has the highest degree of accuracy

$$\int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}(w(\alpha) + w(\beta)).$$

   Find its degree of accuracy.

4. Let I be an interval in $\mathbb{R}$. Consider the scalar Cauchy problem, that is, to find a real-valued function $y \in C^1(I)$ such that

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t_0) = y_0, \end{cases}$$

   where $t_0 \in I$ is a given point, $f(t,y)$ is a given real-valued function.

   (a) (5 pts) Write down the θ method that is used to approximate the problem.

   (b) (10 pts) Analyze the local truncation error for $\theta \neq \frac{1}{2}$.

   (c) (5 pts) Analyze the local truncation error for $\theta = \frac{1}{2}$.

5. Consider the scalar hyperbolic problem

$$\begin{cases} u_t + au_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $a > 0$ is a constant and u_0 is a given function.

(a) (10 pts) Use method of characteristic to determine the solution.

(b) (10 pts) Assume $u_0(x)$ is 2π-periodic, use Von Neumann stability analysis to determine the stability of the following finite difference discretization of the problem

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0.$$

6. Apply the backward Euler method with step size h to the problem $y' = \lambda y$ for a real constant λ. Denote the numerical solution as $y_h(x_k)$, $k \in \mathbb{N}$, where $y_h(x_0) = y(x_0) = y_0$ is the given initial condition of the problem.

(a) (5 pts) Write down explicitly $y_h(x_n)$.

(b) (15 pts) Show that

$$y(x_n) - y_h(x_n) = -\frac{\lambda^2 x_n e^{\lambda x_n}}{2}h + O(h^2).$$