PhD Qualifying Exam in Numerical Analysis

Fall, 2021

1. (15%) Determine the order and stability of the following linear multistep method

$$u_{n+1} = -4u_n + 5u_{n-1} + 2hf_n + 4hf_{n-1}.$$

2. (15%) Consider the two-point boundary value problem

$$-u''(x) = f(x), \quad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

   (a) Find the Green's function for the above problem.

   (b) Show that $\|u\|_\infty \leq \frac{1}{8}\|f\|_\infty$.

3. (20%) Consider the scalar hyperbolic problem

$$u_t + au_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

   where $a > 0$ is a constant and u_0 is a given function.

   (a) Use method of characteristic to determine the solution.

   (b) Assume $u_0(x)$ is 2π-periodic, use Von Neumann stability analysis to determine the stability of the following finite difference discretization of the problem

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}a(u_{j+1}^n - u_{j-1}^n),$$

   where $\lambda = \Delta t/\Delta x$.

4. (20%) A real-valued system of n linear equations in n unknowns consists of a set of algebraic relations and the system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$. Let $A = P - N$ with A symmetric and positive definite. Prove that if the matrix $P + P^{\mathsf{T}} - A$ is positive definite, then

   (a) P is invertible;

   (b) the iterative method defined in $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + P^{-1}\mathbf{r}^{(k)}$, where $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$, is monotonically convergent with respect to norm $\|\cdot\|_A$;

   (c) $\rho(B) \leq \|B\|_A < 1$, where B is the iteration matrix.

5. (15%) Given $n+1$ distinct points $x_0 < x_1 < \cdots < x_n$ and $n+1$ corresponding values $y_0, y_1, \ldots, y_n$. Prove that

(a) there exists a unique polynomial $\Pi_n \in \mathbb{P}_n$ such that $\Pi_n(x_i) = y_i$ for $i = 0, 1, \ldots, n$;

(b) there exists a unique piecewise-polynomial with degree 3, $S(x)$, such that

$$S(x) = \begin{cases} S_0(x) & x \in [x_0, x_1], \\ S_1(x) & x \in [x_1, x_2], \\ \vdots & \vdots \\ S_{n-1} & x \in [x_{n-1}, x_n] \end{cases}$$

satisfies

(1) $S_j(x_j) = y_j$ and $S_j(x_{j+1}) = y_{j+1}$ for $j = 0, 1, \ldots, n-1$;
(2) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for $j = 0, 1, \ldots, n-2$;
(3) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for $j = 0, 1, \ldots, n-2$;
(4) $S''(x_0) = S''(x_n) = 0$.

6. (15%) Let a composite Newton-Cotes formula, with n even, be used. Prove that if $f \in C^{n+2}([a,b])$, then the quadrature error

$$E_{n,m}(f) = \frac{b-a}{(n+2)!} \frac{M_n}{(n+2)^{n+3}} H^{n+2} f^{(n+2)}(\xi),$$

where $\xi \in (a,b)$, $H = \dfrac{b-a}{m}$, and $M_n = \begin{cases} \displaystyle\int_0^n t\, \pi_{n+1}(t)\, dt < 0 & \text{for closed formulae,} \\ \displaystyle\int_{-1}^{n+1} t\, \pi_{n+1}(t)\, dt > 0 & \text{for open formulae,} \end{cases}$

having defined $\pi_{n+1}(t) = \prod_{i=0}^{n} (t-i)$.