National Yang Ming Chiao Tung University
Department of Applied Mathematics

Real Analysis Ph.D. Qualifying Exam, Spring 2021

There are 6 question sets of total 100 points.
Answer questions as carefully and completely as possible.
Do not make formal arguments without mathematical justification.
If you use a major theorem, mention it by name and check its hypotheses.

1. (a) (5 %) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function such that $f'(x)$ exists and $|f'(x)| \le 1$ for almost all $x \in \mathbb{R}$. Is it true that

$$f(b) - f(a) = \int_{[a,b]} f'(x)\,dx \qquad \text{for every closed intervals } [a,b] \subset \mathbb{R}?$$

Justify your answer.

(b) (5 %) Suppose that $g : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at *all* $x \in \mathbb{R}$, Is such a g necessarily of *bounded variation* for every closed interval $[a,b] \subset \mathbb{R}$? Justify your answer.

(c) Let $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \in (0,\infty); \\ 1, & \text{if } x = 0. \end{cases}$

i. (5 %) $f \in L^1([0,\infty))$? Justify your answer.

ii. (5 %) Does the improper Riemann integral $\displaystyle\int_0^\infty f(x)\,dx$ converge? Justify your answer.

2. (10 %) Let $f_k : \mathbb{R} \longrightarrow \mathbb{R}$ be a sequence of Borel measurable functions and let

$$E = \left\{ x \in \mathbb{R} \mid \lim_{k\to\infty} f_k(x) \text{ exists and is finite} \right\}.$$

Is the set E Borel measurable? Justify your answer.

3. (a) Suppose that $f : \mathbb{D} = \{(x,y) \mid x^2 + y^2 \le 1\} \longrightarrow \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2}, & \text{if } (x,y) \ne (0,0); \\ 0. & \text{if } (x,y) = (0,0). \end{cases}$$

i. (4 %) Compute the *iterated integrals* of f over $\mathbb{D}$.

ii. (6 %) Is f Lebesgue integrable over $\mathbb{D}$? Justify your answer.

(b) (5 %) Let $f : [0,1] \longrightarrow \mathbb{R}$ be Lebesgue measurable. Suppose $f(x) - f(y)$ is Lebesgue integrable over the square $[0,1] \times [0,1]$. Is it true that $f \in L^1([0,1])$? Justify your answer.

4. (a) (5 %) Let $f(x) = x^3$ for $x \in [0,1]$. Extend f periodically to be defined for all $x \in \mathbb{R}$ and still denote the extended function by f. Define $f_k(x) = f(kx)$ for $k \in \mathbb{N}$. Show that the sequence $\{f_k\}_k \in L^1([0,1])$.

   (b) (5 %) Is it true that there is an $f \in L^1([0,1])$ so that $\int_{[0,1]} f_k(x) \cdot g(x)\, dx \longrightarrow \int_{[0,1]} f(x) \cdot g(x)\, dx$ for all $g \in L^\infty([0,1])$? Justify your answer.

   (c) (5 %) Does the sequence $\{f_k\}_k$ converges with respect to L^1-norm to some $f \in L^1([0,1])$? Justify your answer.

5. (20 %) Given $0 < \epsilon < 1$. Construct a dense measurable subset $E \subset [0,1]$ such that the outer measure of E is ϵ.

6. (20 %) Let $f(x,y)$, $0 \le x, y \le 1$ satisfy the following conditions: For each x, $f(x,y)$ is an integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx} \int_0^1 f(x,y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\, dy.$$