Discrete Mathematics

(Qualifying Examination for the Ph. D. Program, Spring 2021)

Note: Each of the followings weights 12.5 points. Explain your answers as clear as possible for full credits.

1. In how many ways can one decompose a convex n-gon (labeled) into triangles by n – 3 non-intersecting diagonals.

2. Let c(n, k) denote the number of permutations $\pi \in S_n$ with exactly k cycles. Find c(7, 3).

3. Prove that a simple planar graph is 5-colorable.

4. Let $A_1, A_2, \ldots, A_n$ be n distinct subsets of the set $[n] = \{1, 2, \ldots, n\}$. Prove that there exists an element $x \in [n]$ such that the sets $A_i \setminus \{x\}$, $i \in [n]$, are all distinct.

5. Prove that if a simple graph of order n has more than $n \cdot (n-1)^{1/2}/2$ edges, then G has girth at most 4, that is, G contains a C_3 or a C_4.

6. Let m and n be positive integers, $m < n$. Prove that a Latin square of order n contains a sub-square of order m if and only if $n \geq 2m$.

7. Find a subdivision of $K_{3,3}$ as a subgraph of the Petersen graph. Moreover, use this fact to prove that the Petersen graph is not planar.

8. Let X be a nonempty set and B is a collection of subsets of X such that each subset of B contains at least two elements and any two distinct elements of X are contained in exactly one subset of B. Prove that the cardinality of X is not larger than the cardinality of B.