NATIONAL CHIAO TUNG UNIVERSITY

Real Analysis Ph.D. Qualifying Exam, Fall 2020

1. (6%) Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with

$$\int_0^1 |f_n(x)|^2 dx \leq 1$$

and $f_n \to 0$ a.e. on $[0,1]$. Show that $\int_0^1 |f_n(x)| dx \to 0$.

2. Assume $f : [a,b] \to \mathbb{R}$ has bounded variation.

   (a) (6%) Show that if the function $V(x) = V[a,x]$ is absolutely continuous on $[a,b]$, then f is absolutely continuous on $[a,b]$.

   (b) (6%) Show that if $\int_a^b |f'(x)| dx = V[a,b]$, then f is absolutely continuous on $[a,b]$.

3. (12%) Assume $f \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for some $0 < p < \infty$. Show that $f \in L^q(\mathbb{R}^n)$ for all $q > p$ and

$$\lim_{q \to \infty} \|f\|_q = \|f\|_\infty.$$

4. (10%) Let m be Lebesgue measure on $\mathbb{R}$. Suppose that $A \subseteq \mathbb{R}$ is Lebesgue measurable and that

$$m(A \cap (a,b)) \leq \frac{b-a}{3}$$

for any $a, b \in \mathbb{R}$, $a < b$. Prove that $m(A) = 0$.

5. (10%) Suppose that $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$ and that $f : X \to [0, \infty)$ is a measurable function. Show that $\int_X f d\mu < \infty$ if and only if the series

$$\sum_{n=0}^{\infty} \mu(\{x | f(x) \geq n\})$$

converges.

6. (20%) Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, which converges almost everywhere to a function f in $L^p(\mathbb{R}^n)$. Show that $\{f_n\}$ converges to f in $L^p(\mathbb{R}^n)$ if and only if $||f_n||_{L^p(\mathbb{R}^n)} \to ||f||_{L^p(\mathbb{R}^n)}$ as $n \to \infty$. What can you say when $p = \infty$?

7. (15%) Show that the unit ball $S = \{x \in X | \, ||x|| \leq 1\}$ of a Banach space X is compact if and only if X is of finite dimensional.

8. (15%) Define a maximal function of measurable function f as follows

$$(Mf)(x) = sup_{B \in \mathcal{B}} \frac{1}{|\mu(B)|} \int_B |f| d\mu$$

where $\mathcal{B}$ is a collection of balls centred at x and $|\mu(B)|$ is the Lebesgue measure of B. Assume we know the following weak L^1−estimate:

$$\mu(\{Mg > t\}) \leq \frac{C}{t} ||g||_{L^1(\mathbb{R}^n)}$$

where C is a constant, holds for $g \in L^1(\mathbb{R}^n)$. Show that, there exists $C_1 > 0$

$$||Mf||_{L^p(\mathbb{R}^2)} \leq C_1 ||f||_{L^p(\mathbb{R}^2)}$$

for $f \in L^p(\mathbb{R}^2)$.