QUALIFY EXAM.
PARTIAL DIFFERENTIAL EQUATIONS.
SEPTEMBER 2020

- This exam contains 4 problems with a total 100 points. Solve these 4 problems to get full credits.

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1. (20 points) Let $u = u(x, y, t)$ be a solution of the Cauchy problem

$$
\begin{cases} u_t + u(u_x + u_y) = 0, & \text{for } t > 0, \\ u(x, y, 0) = \dfrac{1}{x^2 + y^2 + 1}. \end{cases} \tag{0.1}
$$

Prove that the Cauchy problem (0.1) has a classical C^1 solution up to the time $T = \min\limits_{(x,y)} \dfrac{(x^2 + y^2 + 1)^2}{2|x + y|}$. Find T explicitly.

– **Hint.** Use the characteristic method and the inverse function theorem.

2. (a) (15 points) Let $U \in C^2(\mathbb{R}^2)$ be a solution to

$$\partial_s \partial_\lambda U = 0 \text{ in } \mathbb{R}^2.$$

Prove that $U(s, \lambda)$ can be written in the form of

$$U(s, \lambda) = F(s) + G(\lambda), \quad \text{for } (s, \lambda) \in \mathbb{R}^2,$$

where F and G are some functions defined on $\mathbb{R}$.

(b) (15 points) Use and only use the results from (a) to reduce the well-known d'Alembert's formula for the solution $u = u(t, x) \in C^2(\mathbb{R}^2)$ to the following equation

$$
\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 & \text{in } (0, \infty) \times \mathbb{R}, \\ u = g, \quad \partial_t u = h & \text{on } \{0\} \times \mathbb{R} \end{cases}
$$

with $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ to be given.

– **Hint.** For (a), one may consider the following change of variables:

$$
\begin{cases} s = x + t, \\ \lambda = x - t. \end{cases}
$$

Define a $C^2(\mathbb{R}^2)$ function by $U(s, \lambda) := u(t, x)$, for all $(t, x) \in \mathbb{R}^2$, then prove that $\partial_t^2 u - \partial_x^2 u = 0$ holds in $\mathbb{R}^2$ if and only if $\partial_s \partial_\lambda U(s, \lambda) = 0$ holds in $\mathbb{R}^2$.

3. (20 points) Take a positive integer $N \geq 3$. Let $B_R(0)$ be a ball centered at 0 with radius $R > 0$ in $\mathbb{R}^N$. Given functions $f \in C^{0,\alpha}(\overline{B_R(0)})$[1], and $g \in C(\partial B_R(0))$.

[1] Here $C^{0,\alpha}(\overline{B_R(0)})$ denotes the Hölder continuous space on a compact ball $\overline{B_R(0)}$ of radius $R > 0$, for some $0 < \alpha < 1$.

Suppose that $u \in C^2(B_R(0)) \cap C(\overline{B_R(0)})$ is a solution of

$$\begin{cases} -\Delta u = f & \text{in } B_R(0), \\ u = g & \text{on } \partial B_R(0). \end{cases}$$

Prove that the following identity holds:

$$\begin{aligned} u(0) = & \frac{1}{|\partial B_R(0)|} \int_{\partial B_R(0)} g(y)\, dS(y) \\ & + \frac{1}{N(N-2)|B_1(0)|} \int_{B_R(0)} \left(\frac{1}{|y|^{N-2}} - \frac{1}{R^{N-2}} \right) f(y)\, dy, \end{aligned}$$

where $|E|$ is the Lebesgue measure of a measurable set $E \subset \mathbb{R}^N$.

4. *(The problem is based on a result due to the work of Brezis-Merle: Uniform estimates and blow-up behavior of solutions of $-\Delta u = V(x)e^u$ in two dimensions.)*

Let $B_R(0)$ be a ball centered at 0 with radius $R > 0$ in $\mathbb{R}^2$. Let $f : \overline{B_R(0)} \to \mathbb{R}$ satisfying $f \in C^{0,\alpha}(\overline{B_R(0)})$. Let $u \in C^2(B_R(0)) \cap C(\overline{B_R(0)})$ be the solution of

$$\begin{cases} -\Delta u = f & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0). \end{cases}$$

Answer the following questions:

(a) (10 points) Define $U : \mathbb{R}^2 \to \mathbb{R}$ by

$$U(x) := \frac{1}{2\pi} \int_{B_R(0)} \log \left(\frac{2R}{|x-y|} \right) |f(y)|\, dy, \quad \text{for } x \in \mathbb{R}^2.$$

Prove that $|u(x)| \le U(x)$ for all $x \in \overline{B_R(0)}$.

(b) (10 points) Prove that for any $0 < \lambda < 2$, we have

$$\int_{B_R(0)} \frac{1}{|x-y|^\lambda}\, dx \le \int_{B_R(0)} \frac{1}{|x|^\lambda}\, dx, \quad \text{for any } y \in B_R(0).$$

(c) (10 points) By using the conclusions of (a) and (b), prove that for any $\mu \in (0, 4\pi)$,

$$\int_{B_R(0)} \exp \left(\frac{(4\pi - \mu)|u(x)|}{\|f\|_{L^1(B_R(0))}} \right) dx \le \frac{4\pi^2}{\mu} (2R)^2$$

holds.

- **Hint.** For (a), use the maximum (or comparison) principle for subharmonic functions. For (c), you may use the Jensen's inequality from Real Analysis: Let $F : \mathbb{R} \to (0, \infty)$ be a convex function, and $w : \overline{B_R(0)} \to [0, \infty)$ be a L^1-function with $\int_{B_R(0)} w(y)\, dy = 1$. Then one has

$$F \left(\int_{B_R(0)} \varphi(y) w(y)\, dy \right) \le \int_{B_R(0)} w(y) F(\varphi(y))\, dy,$$

for any $\varphi : \overline{B_R(0)} \to [0, \infty)$ with $\int_{B_R(0)} \varphi(y) w(y)\, dy < \infty$. Try to choose some special F, φ and w to answer (c).