PhD Qualifying Exam in Numerical Analysis

Fall, 2020

1. (15%) Let

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

   Suppose we want to solve linear system $Ax = b$.

   (a) Write down the matrix splitting of the Gauss-Seidel (GS) iterative method.

   (b) Prove that the GS method can always find the answer x for any initial guess and any right hand side b.

   (c) If the matrix A inherits a small perturbation, i.e. $A := A + \epsilon E$ where $\epsilon \ll 1$, will the GS method still converge for any initial guess? If "yes", justify your answer. Otherwise, give a counter example.

2. (15%) Consider an underdetermined linear system $Ax = b$ where

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

   (a) Find the pseudo inverse A^+ of the matrix A.

   (b) Show that $I - (A^+ \cdot A)$ gives the projection of x onto the null space of A.

   (c) Suppose the matrix A inherits a small noise from inputs. Let x and x_ϵ be the pseudo solutions of the original system $Ax = b$ and the noisy system $A_\epsilon x = b$, respectively, where $A_\epsilon = A + \epsilon E$, E is a unit random matrix and $\epsilon \ll 1$. Will the solution still close to the original answer x for arbitrary right hand side? If "yes", justify your answer. Otherwise, give a counter example.

3. (20%) Consider

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix}.$$

   (a) Evaluate the singular value decomposition of A.

   (b) Find a matrix B which is the best rank 2 approximation to A in the 2-norm and specify $\|A - B\|_2$.

4. (10%) Determine x_0 and x_1 so that the following quadrature rule has the highest degree of precision.

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}\left(f(x_0) + f(x_1)\right).$$

   Find its degree of precision.

5. (20%) Consider the following method for solving the heat equation $u_t = u_{xx}$:

$$U_i^{n+2} = U_i^n + \frac{2k}{h^2}\left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right),$$

where k is the time step size and h is the spatial step size.

(a) Determine the order of accuracy of this method (in both space and time).

(b) Determine the stability of this method.

6. (20%) Consider gradient descent method for finding the minimum of $f(x) : R^n \to R$

(a) Explain briefly the algorithm

(b) Show that, with optimal step size, i.e., every steps it achieves the minimum in the descent direction, the gradient method is zig-zagging, i.e., $\nabla f(x_{n+1}) \cdot \nabla f(x_n) = 0$.