NATIONAL CHIAO TUNG UNIVERSITY

2020 Real Analysis Spring Ph.D. Qualifying Exam

1. (15 %) Prove the followig statement: If $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions which converge to a real-valued function f a.e. on a measurable set Ω of finite measure, then, given $\eta > 0$, there is a subset $A \subset \Omega$ with $|A| < \eta$ such that f_k converges to f uniformly on $\Omega \setminus A$.

2. (15 %) Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions which converge to f in measure on a measurable set Ω and suppose there is an integrable function g satisfying $|f_k(x)| \leq g(x)$ for any k and $x \in \Omega$, prove f_k converges to f in $L^1(\Omega)$ space.

3. (5 %) Suppose $\alpha > 1$ and $\{x_k\}_{k=0}^{\infty}$ is a non-negative sequence satisfying

$$\begin{cases} x_0 \leq 2^{\frac{-\alpha}{(\alpha-1)^2}}, \\ x_{k+1} \leq 2^{k\alpha} x_k^{\alpha} \quad \text{for any } k \geq 0, \end{cases} \tag{1}$$

is the statement $\lim_{k\to\infty} x_k = 0$ true? Why?

4. (5 %) Suppose $f, g \in L^p([0,1])$, $p > 2$, $\ell > 1$, and

$$\|f\|_{L^p([0,1])}^2 + \ell \|f\|_{L^2([0,1])}^2 \leq \|fg\|_{L^1([0,1])}.$$

Prove that there is a constant c independent of ℓ such that

$$\|f\|_{L^r([0,1])} \leq c\, \ell^{\frac{-2(p-r)}{r(p-2)}} \|g\|_{L^{r'}([0,1])} \quad \text{where } r \in [2,p] \text{ and } \frac{1}{r} + \frac{1}{r'} = 1.$$

5. (10 %) Suppose $\delta > 0$ and $F \subset [0,1]^3$ is an open set, describe a procedure to produce a collection of disjoint dyadic subcubes $\{Q_k\}_{k=1}^{\infty}$ of $[0,1]^3$ satisfying

$$\begin{cases} |F \cap \check{Q}_k| \leq \delta |\check{Q}_k| \\ |F \cap Q_k| > \delta |Q_k| \quad \text{for } k \geq 1, \\ |F \setminus \bigcup_{k=1}^{\infty} Q_k| = 0 \end{cases} \tag{2}$$

where $\check{Q}_k$ denotes the dyadic "parent" of Q_k (i.e., Q_k is one of the 2^3 cubes obtained by bisecting the sides of $\check{Q}_k$).

6. (12 %) Let X be a normed linear space. Show that a linear functional T on X is bounded if and only if the kernel of T is closed.

7. (12 %) Let X be a Banach space and $\mathcal{F}$ be a family of bounded linear operators from X to a normed space Y. Suppose that for each $x \in X$, there is a constant M_x such that

$$||Tx|| \leq M_x \quad \text{for all } T \in \mathcal{F}.$$

Show that there exists M such that

$$||T|| \leq M \quad \text{for all } T \in \mathcal{F}.$$

8. (14 %) Let S be a subspace of $L^2([0,1]) = \{f \mid f : \mathbb{R} \to \mathbb{R}, \int_0^1 |f|^2 dx < +\infty\}$. Suppose there is a constant K such that

$$|f(x)| \leq K\|f\| \quad \text{for all} \quad x \in [0,1].$$

Show tha the dimesion of S is at most K^2.

9. (12 %) If $f \in L^2([0,2\pi])$, show that

$$\lim_{k \to \infty} \int_0^{2\pi} f(x) \cos kx dx = 0.$$

Prove that the same is true if $f \in L^1[0,2\pi]$