Qualifying Exam.
Ordinary Differential Equations February 2020

This exam. contains 5 problems with a total of 100 points. Do all 5 problems. Show all your work to get full credits.

1. Consider the predator-prey system

$$\begin{cases} x' = \gamma x\left(1 - \dfrac{x}{K}\right) - \alpha x y \\ y' = y\left(\beta x - d\right), \quad \text{constants } \gamma, K, \alpha, \beta, d > 0 \\ x(0) > 0, \ y(0) > 0. \end{cases}$$

(8 points) (a) Show that the solutions $(x(t), y(t))$ are positive for all $t > 0$.
(7 points) (b) Show that the solutions $(x(t), y(t))$ are bounded for all $t > 0$.
(5 points) (c) Show that the solutions $(x(t), y(t))$ exist for all $t > 0$.

2. Consider $x' = Ax$, $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$. Let A be a 4×4 real matrix and

$$A \sim B = \begin{bmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix}$$

(i.e., A is similar to B) with real constants $a > 0$, $b > 0$.

(8 points) (a) If a/b is a rational number, prove that every nontrivial solution $x(t)$ to $x' = Ax$ is periodic and find the period of each individual nontrivial solution $x(t)$.

(7 points) (b) If a/b is an irrational number, prove that every nontrivial solution $x(t)$ satisfies

$$M \leq |x(t)| \leq N$$

for suitable constants M, $N > 0$. Moreover, there exists a nontrivial solution $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$, $x_i(t)$ not identically zero for $i = 1, 2, 3, 4$, such that $x(t)$ is nonperiodic.

(5 points) (c) If a/b is irrational, are all nontrivial solutions $x(t)$ nonperiodic? Give a proof or a counterexample.

3. Consider the nonlinear periodic system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 0 & h(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad h(t) = \frac{\cos(t) + \sin(t)}{2 + \sin(t) - \cos(t)} \tag{1}$$

where $h(\cdot)$ has period $T = 2\pi$.

(8 points) (a) Find a fundamental matrix $\Phi(t)$ of (1). (Hint. Solve $x_2(t)$ first. Then solve $x_1(t)$ by the method of integrating factors.)

(8 points) (b) Find a 2π periodic matrix $P(t)$ and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$. Find the characteristic exponents of (1).

(4 points) (c) Are all solutions of (1) periodic? If not, find all period solutions of (1).

(Please turn this page over and continue with Problems 4 and 5, Page 2.)

4. (20 points) Consider the nonlinear spring motion with friction

$$x''(t) + f(x)x'(t) + g(x) = 0,$$

where $f, g \in C$, $f(x) > 0$, $x \neq 0$, $xg(x) > 0$, $x \neq 0$, $g(0) = 0$, and $G(x) = \int_0^x g(s)ds \to \infty$ as $|x| \to \infty$. Prove that $(x(t), x'(t)) \to (0, 0)$ as $t \to \infty$. (**State the theorem used precisely.**)

5. (5 points) (a) State Dulac's criterion for the system on the xy-plane

$$\begin{cases} x' = F(x, y) \\ y' = G(x, y). \end{cases}$$

(15 points) (b) Apply Dulac's criterion to show that the nonlinear system

$$\begin{cases} x' = -y + x^2 - xy \\ y' = x + xy \end{cases}$$

has no periodic orbit on the xy-plane by using $h(x, y) = (1 + x)^{-m} (1 + y)^{-n}$ for some positive integers m, n to be determined.)