2019 Ph.D. Qualifying Exam in Partial Differential Equations, September 10, 2019.

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Instruction: Please answer all of the following questions. Please justify all the steps in your arguments with clear and rigorous mathematical arguments. In the case when you need to use a specific mathematical theorem to justify your answer, please specify exactly which theorem or theory you are using.

Problem 1(*12 points*) Let $N \geq 2$ be an integer, consider u $\in C^{\infty}(\mathbb{R}^N)$ be a solution to the Laplace equation on $\mathbb{R}^N$, so that $\triangle u = 0$ holds on $\mathbb{R}^N$. Suppose further that u satisfies the following property

$$\int_{\mathbb{R}^N} \left|u(x)\right|^2 \, dx < \infty.$$

Prove that we have $u = 0$ on $\mathbb{R}^N$.

Problem 2(*6 points*) Let Ω be a bounded domain in $\mathbb{R}^N$, for some integer $N \geq 2$. Consider a scalar-valued function u $C^0(\overline{\Omega}) \cap C^{\infty}(\Omega)$ which satisfies $\triangle u = 0$ on Ω. Prove that there exists some absolute constant $C_0 > 0$, *which depends only on* N, such that u satisfies the following apriori estimate for any $x \in \Omega$.

$$d_x \left|\nabla u(x)\right| + d_x^2 \left|\nabla^2 u(x)\right| \leq C_0 \|u\|_{L^{\infty}(\Omega)},$$

where $d_x = dist(x, \partial\Omega)$, which is the distance of x from the boundary of Ω, and that $|\nabla^2 u| = \max\{|\partial_i \partial_j u| : 1 \leq i, j \leq N\}$.

Problem 3(*12 points*) Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, for some integer $N \geq 2$. Consider a scalar-valued function $u \in C^2(\overline{\Omega})$ which satisfies $\triangle u = u^3$ on Ω. Suppose further that $\left|\nabla u(x)\right| \leq 1$ holds for all $x \in \partial\Omega$. Prove that $\left|\nabla u(x)\right| \leq 1$ holds for all $x \in \Omega$.

Problem 4 Let Ω be a bounded domain with a C^1-boundary $\partial\Omega$ in $\mathbb{R}^N$, for some integer $N \geq 2$. Let $T > 0$ be a given positive quantity. Consider now a scalar-valued function $u : [0, T] \times \overline{\Omega} \to \mathbb{R}$, which is twice continuously differentiable spatially on $[0, T] \times \overline{\Omega}$, as well as one time continuously differentiable in the temporal direction on $[0, T] \times \overline{\Omega}$. In order words, we suppose that $\nabla^2 u \in C^0([0, T] \times \overline{\Omega})$ and that $\partial_t u \in C^0([0, T] \times \overline{\Omega})$. Suppose further that for some prescribed continuous function $f \in C^0([0, T] \times \overline{\Omega})$, u is a solution to the following equation on $[0, T] \times \overline{\Omega}$

$$\partial_t u - \triangle u = f. \tag{0.1}$$

Assume in addition that $u(t, x) = 0$ holds for all $(t, x) \in [0, T] \times \partial\Omega$.

(Part a) *(10 points)*Under all the above hypothesis, use integration by parts technique to prove that the following estimate holds for any $\mu \in (0,T]$, where $C_0 > 0$ is some absolute constant *which depends only on N*

$$\begin{aligned}&\int_0^{\mu}\int_{\Omega}|\nabla u(t,x)|^2+|u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega}|u(\mu,x)|^2\,\mathrm{d}x\\ \le & C_0\left\{(1+t_0)\int_{\Omega}|u(0,x)|^2\,\mathrm{d}x+(1+t_0^2)\int_0^{\mu}\int_{\Omega}|f(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right\}.\end{aligned} \tag{0.2}$$

(Part b) *(5 points)*For a given $f \in C^0([0,T]\times\overline{\Omega})$ and *with respect to a prescribed initial profile* $u(0,\cdot)=u_0 \in C^2(\overline{\Omega})$, prove that there is at most one solution u to equation (0.1) on $[0,T]\times\overline{\Omega}$, with $\nabla^2 u \in C^0([0,T]\times\overline{\Omega})$ and $\partial_t u \in C^0([0,T]\times\overline{\Omega})$, which at the same time satisfies the boundary condition $u|_{[0,T]\times\partial\Omega}=0$.

Problem 5 In this problem, we will look at the space-time $\{(t,x): t\in\mathbb{R}, x\in\mathbb{R}\}$, with t stands for the time-variable, and x stands for the spatial variable. Consider a parabolic space-time region Ω, which is defined as follows

$$\Omega=\{(t,x): |x|<2, x^2-4<t<0\}.$$

Note that $\partial\Omega$ consists of two parts, namely Γ and L which are defined as follows

$$\begin{aligned}\Gamma&=\{(x^2-4,x): |x|\le 2\}\\ L&=\{(0,x): |x|\le 2\},\end{aligned}$$

so that $\partial\Omega=\Gamma\cup L$. Under this setting, we consider now a scalar-valued *non-constant* function $u\in C^0(\overline{\Omega})$, with $\partial_x^2 u, \partial_t u \in C^0(\Omega\cup L)$, which is a solution to the following parabolic-type equation on $\Omega\cup L$

$$\partial_t u-\partial_x^2 u+\alpha u=f, \tag{0.3}$$

where the functions $\alpha, f\in C^0(\overline{\Omega})$ are prescribed. Under the above setting and hypothesis, answer the following two questions.

(Part a)*(4 points)* If it happens that $\alpha(t,x)>0$ holds for all $(t,x)\in\overline{\Omega}$, and that $f(t,x)\ge 0$ holds for all $(t,x)\in\overline{\Omega}$, prove that the *strict* negative minimum of u (provided that exists) must take place on Γ.

(part b) *(6 points)* In the case when $\alpha>0$ on $\overline{\Omega}$ and $f\le 0$ on $\overline{\Omega}$, can you draw a conclusion which is analog to that in the previous part. Please also prove your answer.

Problem 6*(15 points)* Let $0<T\le+\infty$. Consider a scalar-valued function $u\in C^1([0,T)\times\mathbb{R})$ which satisfies the following first order equation on $[0,T)\times\mathbb{R}$

$$\partial_t u(t,x)+\partial_x\{f(u(t,x))\}=0, \tag{0.4}$$

where $f\in C^1([0,T)\times\mathbb{R})$ is prescribed. Suppose that there are $x_1,x_2\in\mathbb{R}$ with $x_1<x_2$ such that $f'(u(0,x_1))>f'(u(0,x_2))>0$. Is it possible that $T=+\infty$. If your answer is affirmative, please prove your answer. If your answer is negative, please prove your answer.

Problem 7 Consider the unit ball $B_O(1)=\{x\in\mathbb{R}^3: |x|<1\}$ in $\mathbb{R}^3$. Let $\rho:\overline{B_O(1)}\to(0,\infty)$ be a positive-valued continuous function as defined on the closed ball $\overline{B_O(1)}$. Consider the function $F:\mathbb{R}^3\to\mathbb{R}$ which is defined as follows

$$F(x)=\int_{\{y\in\mathbb{R}^3:|y|\le 1\}}\frac{\rho(y)}{|x-y|}\,\mathrm{d}y. \tag{0.5}$$

With the above background, please answer the following question
(Part a) *(6 points)* For the function F as defined in (0.5), prove that F is smooth on $\mathbb{R}^3 - \overline{B_O(1)}$. That is prove that $F \in C^\infty(\mathbb{R}^3 - \overline{B_O(1)})$.
(Part b) *(4 points)* By means of your result as obtained in *(Part a)*, prove that $\triangle F = 0$ holds on $\mathbb{R}^3 - \overline{B_O(1)}$.
(Part c) *(8 points)* Prove that there exists a constant $C_0 > 0$, which depends only on the density function ρ, such that the following estimate holds for all $x \in \mathbb{R}^3$ with $|x| \geq 2$.

$$0 < F(x) \leq \frac{C_0}{|x|}. \tag{0.6}$$

(Part d) *(2 points)* In the case of $\rho = 1$ on $\overline{B_O(1)}$, do you agree that F, as defined in (0.5) , is indeed radially symmetric on $\mathbb{R}^3 - \overline{B_O(1)}$. Please explain your reason very briefly.

(Part e) *(4 points)* By means of results and observations as obtained in the previous parts, prove that if $\rho = 1$ holds on $\overline{B_O(1)}$, then, we have the following identity which holds for all $x \in \mathbb{R}^3 - \overline{B_O(1)}$.

$$F(x) = \frac{A_0}{|x|}, \tag{0.7}$$

where $A_0 > 0$ is some constant.
(Part f) *(6 points)* Compute the positive constant $A_0 > 0$ which appears in identity (0.7), and express your answer in terms of $\pi = 3.14159......$ Please prove your answer.