Qualifying Examination: Ordinary Differential Equations **September 10, 2019**

This exam. contains 5 problems with a total of 100 points.

1. Consider the equation $\dot{x} = f(t,x)$, $f(t,x) \in C^1$, $|f(t,x)| \leq \phi(t)|x|$ for all t, x, and $\int^\infty \phi(t)dt < \infty$.

(10 points) (a) Prove that the every solution approaches a constant as $t \to \infty$.

(10 points) (b) If, in addition,

$$|f(t,x) - f(t,y)| \leq \phi(t)\,|x-y| \quad \text{for all } x, y.$$

Prove that there is a one to one correspondence between the initial values and the limit values of the solution.

2. Let $\Phi(t)$ be a fundamental matrix of

$$\frac{dx}{dt} = A(t)x$$

and $\varphi(t,\tau,\xi)$ be the solution of the IVP

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + g(t), \\ x(\tau) = \xi, \end{cases} \tag{1}$$

where $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$ and $g(t) \in \mathbb{R}^n$ are continuous.

(5 points) (a) Find a constant matrix $\Lambda \in \mathbb{R}^{n\times n}$ such that

$$\varphi(t,\tau,\xi) \equiv \Phi(t)\Lambda\xi + \int_\tau^t \Phi(t)\Phi^{-1}(\eta)g(\eta)d\eta \tag{2}$$

is the solution of (1).

(5 points) (b) With the constant matrix Λ you find in (a), prove that $\varphi(t,\tau,\xi)$ in (2) is the solution of (1).

(5 points) (c) When $A(t) \equiv A$ is a constant matrix, solve the IVP

$$\begin{cases} \dfrac{dx}{dt} = Ax + g(t), \\ x(\tau) = \xi. \end{cases}$$

(5 points) (d) Solve the forced harmonic oscillator initial value problem

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = a, \quad y'(0) = b. \end{cases} \tag{3}$$

Hint to (d). Write the equation in (3) as the nonhomogenious system

$$\begin{cases} y_1'(t) = -y_2 \\ y_2'(t) = y_1 + f(t). \end{cases}$$

(Please turn this page over and continue with Problems 3, 4 and 5, Page 2.)

3. Consider the equation $\dot{y} = (a\cos t + b)y - y^3$ in $\mathbb{R}$, where $a, b > 0$.

(8 points) (a) Study the local stability of the zero solution.

(12 points) (b) Suppose it is known there is a unique 2π-periodic solution $y_p(t;\xi_0)$, where $y_p(t;\xi_0) = \xi_0 \in [c, a+b+1]$, $c > 0$. Show that $y_p(t;\xi_0)$ is locally stable.

4. (7 points) (a) State the Poincaré-Bendixson Theorem for the autonomous system

$$\mathbf{x}' = f(\mathbf{x}),\ \mathbf{x} \in D \subset \mathbb{R}^n.$$

(13 points) (b) Apply the Poincaré-Bendixson Theorem to show that the nonlinear system

$$\begin{cases} x' = x + y - x(x^2 + 2y^2) \\ y' = -x + y - y(x^2 + 2y^2). \end{cases} \tag{4}$$

has a limit cycle. **Hint.** Convert the system (4) from rectangular coordinates to polar coordinates.

5. (15 points) (a) Prove that the origin is asymptotically stable of the 3-dimensional system

$$\begin{cases} x_1' = -2x_2 + x_2x_3 - x_1^3 \\ x_2' = x_1 - x_1x_3 - x_2^3 \\ x_3' = x_1x_2 - x_3^3. \end{cases} \tag{5}$$

Note that you are requested to state completely the name and content of the theorem which you apply. You need not to give a proof of the theorem.

(5 points) (b) Is the origin a sink to (5)?