PhD Qualifying Exam in Numerical Analysis

Fall, 2019

(Total of 150 points.)

1. (10 pts) A preconditioner P of a matrix A is a matrix such that $P^{-1}A$ has a smaller condition number than A. For the linear system $\begin{bmatrix} 1 & 2 \\ 0 & 10^{-20} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 10^{-20} \end{bmatrix}$, can you make it better conditioned? What is your preconditioner for this system? Compare the condition numbers between the old A and new $P^{-1}A$ systems.

2. Consider the problem $u''(x) = \lambda u$ for all $x \in (0,1)$, $u'(0) = \sigma$, and $u(1) = 0$, where λ and σ are constants.

   (A) (10 pts) For $\lambda = 0$, find an exact solution $u(x)$ of the problem, if it exists. For $\lambda \neq 0$, find an exact solution $u(x)$ of the problem, if it exists.

   (B) (20 pts) For $\lambda = 0$ and $\sigma = 0$, find a finite-difference (FD) approximation $u_h(x)$ of $u(x)$ with $h = 1/4$. Determine the error $\|u_h(x) - u(x)\|_\infty$ of your $u_h(x)$ with $h = 1/4$. Determine and prove the convergence order of your FD, i.e., determine α in $\|u_h(x) - u(x)\|_\infty = O(h^\alpha)$.

3. Consider the linear system $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$,

   $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

   (A) (10 pts) Jacobi's Method (JM) written in component form is $x_i^{(k)} = (b_i - \sum_{j=1, j\neq i}^{N} a_{ij} x_j^{(k-1)})/a_{ii}$. Solve the system by JM to the first 2 iterates and fill your answers in the following table.

Use rational numbers such as $\frac{1}{3}$ NOT 0.333 in the table.

Table. JM Iteration			
k	0	1	2
$x_1^{(k)}$	0		
$x_2^{(k)}$	0		
$x_3^{(k)}$	0		

(B) (10 pts) Write an algorithm (pseudo code) for JM.

(C) (10 pts) Show that JM in matrix form is $\vec{x}^{(k)} = -D^{-1}(L+U)\vec{x}^{(k-1)} + D^{-1}\vec{b}$, $k = 1, 2, 3, \cdots$. What are D, L, and U?

3. Minimization problems:

(A) (10 pts) The change of the function (height) $f(\vec{x})$ at $\vec{x} = (x, y) \in R^2$ in the direction $\vec{p} \in R^2$ is a directional derivative defined as $D_{\vec{p}} f(\vec{x}) = \lim_{t\to 0} \dfrac{f(\vec{x} + t\vec{p}) - f(\vec{x})}{t}$, where $\vec{p} = (p_1, p_2)$ a given unit vector (direction). Show that the minimum value of $D_{\vec{p}} f\ (\vec{x})$ is $-|\nabla f(\vec{x})|$ and $\vec{p} = -\nabla f(\vec{x}) / |\nabla f(\vec{x})|$, where ∇ is the gradient operator.

(B) (20 pts) The method of gradient (steepest) descent is an iterative process $\vec{x}_k = \vec{x}_{k-1} + \alpha_{k-1}\vec{p}_{k-1}$ of changing $\vec{x}_{k-1} = (x_{k-1}, y_{k-1})$ by stepping a length α_k in the gradient direction $\vec{p}_{k-1}$. Show that the gradient vector $\nabla f(\vec{x}_0)$ is *perpendicular* to the tangent vector $\vec{r}\,'(t_0)$ to the level curve $f(\vec{x}) = c$ at $\vec{x} = \vec{x}_0 = (x_0, y_0) = \langle x(t_0), y(t_0)\rangle = \vec{r}(t_0)$, where $\vec{r}\,'(t) = \frac{d}{dt}\langle x(t), y(t)\rangle$. Draw a graph with a mountain, surface, level curve, tangent vector, gradient vector, valley, and all math notations.

(C) (10 pts) Consider the minimization problem: Minimize $z = f(\vec{x}) = \frac{x^2}{4^2} + y^2$, $\forall \vec{x} = \langle x, y\rangle \in R^2$. Use the gradient descent method to find the minimum value of this problem with $\vec{x}_0 = \langle 2, \frac{1}{4}\rangle$ and $\alpha_k = \frac{\sqrt{5}}{4}\ \forall k$.

(D) (10 pts) Show that $\vec{x}^*$ minimizes $\phi(\vec{x}) \iff A\vec{x}^* = \vec{b}$, where $A \in R^{N\times N}$ is a symmetric and positive definite matrix and $\phi(\vec{x}) = \dfrac{1}{2}\vec{x}^T A\vec{x} - \vec{x}^T\vec{b}$. (Hint: $h(\alpha) = \phi(\vec{x} = +\alpha\vec{p})$.)

4. Newton's method finds successively approximations to a root (unknown solution) x^* of a nonlinear equation

$$g(x) = 0, \tag{1}$$

i.e., it iteratively solves the linearized equation

$$\begin{aligned} g'(x^{(0)})w &= g(x^{(0)}), \quad w = x^{(0)} - x^{(1)}, & (2) \\ g'(x^{(0)})w &= \lim_{t \to 0} \frac{g(x^{(0)} + tw) - g(x^{(0)})}{t}, & (3) \end{aligned}$$

where $x^{(1)}$ is the next iterate (unknown) to be solved with a given $x^{(0)}$, then $x^{(2)}$ is solved with $x^{(1)}$, and so on.

(A) (10 pts) Under what conditions, show that the rate of convergence of Newton's method is quadratic, i.e., $\left|x^{(n)} - x^*\right| \leq c \left|x^{(n-1)} - x^*\right|^2$. What is c?

(B) (10 pts) For the coupled nonlinear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = f_1(x_1, x_2) \\ a_{21}x_1 + a_{22}x_2 = f_2(x_1, x_2) \end{cases} \tag{4}$$

written in the matrix form $AX = F$ with two unknown solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X$, the linear operator $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (a matrix), and two nonlinear functions $\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = F$, derive the linearized system of (4) in matrix form that corresponds to (2) and (3).

(C) (10 pts) For the nonlinear differential equation (DE)

$$-u''(x) = f(u(x)) = e^{u(x)} \tag{5}$$

with an unknown solution $u(x)$, the positive linear operator $-\frac{d^2}{dx^2}$, and the nonlinear functional $f(u)$, derive the linearized DE of (5) that corresponds to (2) and (3). Show that $-\frac{d^2}{dx^2}$ is linear.