Qualifying Examination: Ordinary Differential Equations February 21, 2019

1. Let $\varphi(t)$ be the solution of

$$\begin{cases} x'(t) = f(t, x(t)) \in C, & t_0 \le t \le b \quad (b > t_0) \\ x(t_0) = x_0. \end{cases}$$

Let $x(t)$ be a scalar, differentiable function satisfying

$$\begin{cases} x'(t) \le f(t, x(t)), & t_0 \le t \le b \\ x(t_0) \le x_0. \end{cases}$$

(15 points) Is it true that $x(t) \le \varphi(t)$ on $[t_0, b]$? Give a proof or a counterexample.

2. Consider

$$x' = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x, \tag{1}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real, constant, nonsingular matrix. Suppose that the origin is a center of linear system (1) and $x = \varphi(t)$ is a nontrivial (real) solution.

(20 points) Find the equation of $x = \varphi(t)$ and prove that the trajectory of $x = \varphi(t)$ lies on an ellipse on the x_1x_2-plane. In addition, give a necessary and sufficient condition on matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the trajectory of $x = \varphi(t)$ lies on a circle on the x_1x_2-plane. (Justify your answer.)

3. Consider the linear periodic system

$$x' = A(t)x, \quad A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}, \tag{2}$$

where $A(t)$ is continuous on $\mathbb{R}$ and is periodic with period T, i.e., $A(t) = A(t+T)$ for all t.

(15 points) (a) If $\Phi(t)$ is a fundamental matrix of (2), prove that there exists $P(t) \in \mathbb{C}^{n\times n}$ which is nonsingular and satisfies $P(t) = P(t+T)$ and there exists $R \in \mathbb{C}^{n\times n}$ such that $\Phi(t) = P(t)e^{tR}$. Moreover, the nonautonomous linear system (2), under the linear change of coordinates $x = P(t)y$, reduces to the autonomous linear system

$$y' = Ry.$$

(10 points) (b) Let $\Phi(t)$ be a fundamental matrix of (2). Prove that, for any positive integer m, (2) has a nontrivial mT-periodic solution if and only if $(\Phi^{-1}(0)\Phi(T))^m$ has one as an eigenvalue.

(Please turn this page over and continue with Problems 4 and 5, Page 2.)

4. (5 points) (a) Let γ be a periodic orbit of period T of the autonomous system

$$x' = f(x) \in C^1, \quad f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n. \tag{3}$$

Describe briefly what is the Poincaré map P in determining the orbital stability of periodic orbit γ for (3).

(15 points) (b) Consider the nonlinear system

$$\begin{cases} x' = -y + x(1 - x^2 - y^2) \\ y' = x + y(1 - x^2 - y^2). \end{cases} \tag{4}$$

(i) Let the transverse section Σ be the ray $\theta = \theta_0$ through the origin. Find the Poincaré map P for any point on Σ. Find a periodic orbit in the xy-plane.

(ii) Show that nonlinear system (4) has exactly one periodic orbit in the xy-plane. Find the periodic orbit and its period. Determine the stability of the periodic orbit.

5. (8 points) (a) State (i) the Liapunov stability and asymptotic stability theorem and (ii) the Liapunov instability theorem for the IVP

$$x' = f(x), \quad x(0) = x_0, \ x \in D \subset \mathbb{R}^n.$$

(12 points) (b) Discuss the (asymptotic) stability of the origin of the 3-dimensional system

$$\begin{cases} x' = -2y + yz \\ y' = x - xz \\ z' = xy. \end{cases}$$

Hint. Find a suitable Liapunov function $V(x, y, z)$.