PhD Qualify Exam in Numerical Analysis

Fall 2018

(1) (15%) Derive a method to compute a real root of polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$. Discuss the convergence of your method. How to compute the next real root of $p(x)$.

(2) (10%) Let A be a real symmetric matrix. Prove that A is positive definite if and only if all eigenvalues of A are positive.

(3) (20%) Compare Gradient method with Conjugate Gradient method for solving the linear system $Ax = b$ with symmetric positive definite A.

(4) Consider the model problem

$$\begin{cases} -u''(x) + 2u(x) = (2 + \pi^2) \sin \pi x, \text{ for } 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

(a) (10%) Using the centered difference method with uniform mesh to discretize (1), please derive the coefficient matrix A and right hand side vector b of the associated linear system $Ax = b$.

(b) (15%) Show that the coefficient matrix A is not only strictly diagonal dominant but also symmetric positive definite.

(5) (10%) A clamped cubic spline s for a function f is defined by

$$s(x) = \begin{cases} s_0(x) = 1 + Bx + 2x^2 - 2x^3, & \text{if } 0 \leq x < 1, \\ s_1(x) = 1 + b(x-1) - 4(x-1)^2 + 7(x-1)^3, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Find $f'(0)$ and $f'(2)$.

(6) (10%) Give an iterative method to solve the linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 2x_1 + 2x_2 + 2x_3 &= 4, \\ -x_1 - x_2 + 2x_3 &= -5. \end{aligned}$$

and prove the convergence of the giving method.

(7) (10%) Find the constants c_0, c_1 and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.