Qualifying Examination: Ordinary Differential Equations September 13, 2018

This examination contains 5 problems with a total of 100 points.
Show all your work to get full credits.

1. (8 points) (a) Solve solution $x(t,a)$ of the IVP

$$\begin{cases} x'(t) = x^2 \\ x(0) = a, \end{cases} \tag{1}$$

where a is real parameter, and discuss the maximal interval I_a of existence of solution $x(t,a)$.

(12 points) (b) Define the Picard iteration for problem (1). Do you expect the Picard iteration converges to $x(t,a)$ on the maximal interval I_a? Find interval $\tilde{I}_a$ in which the Picard iteration converges to $x(t,a)$ for $t \in \tilde{I}_a$.

2. Consider the second order inhomogeneous equation

$$u''(t) + u(t) = \cos \omega t, \quad \text{constant } \omega > 0. \tag{2}$$

(10 points) (a) Suppose $\omega \neq 1$, show that (2) has a unique periodic solution of period $2\pi/\omega$.

(10 points) (b) Suppose $\omega = 1$, show that (2) has no periodic solutions of period 2π. Find the general solution of (2) and show the resonance phenomena.

3. (20 points) Find $\alpha \in \mathbb{R}$ such that the planar system

$$\begin{cases} x' = y \\ y' = \alpha(1 - x^2 - y^2)y - x \end{cases}$$

has an orbitally asymptotically stable periodic orbit. State the theorem used.

4. (20 points) Verify that the origin is asymptotically stable and its basin of attraction contains the unit disk for the system

$$\begin{cases} x' = y \\ y' = -x - (1 - x^2)y. \end{cases}$$

(Basin of attraction for the origin := the set of all initial points that flow to the origin as $t \to \infty$.) (**Hint.** Try a Lyapunov function $V = ax^2 + bxy + cy^2$.)

5. (7 points) (a) State the Poincaré-Bendixson Theorem for the autonomous system

$$\mathbf{x}' = f(\mathbf{x}).$$

(13 points) (b) Apply the Poincaré-Bendixson Theorem to show that the equation

$$x'' + \beta\left[x^2 + (x')^2 - 1\right]x' + x^3 = 0$$

(β is a positive constant) has at lest one periodic solution inside the annular region

$$\left\{(x,y) \mid 1 \le x^4 + 2y^2 \le 2\right\}.$$