NATIONAL CHIAO TUNG UNIVERSITY

2018 Real Analysis Ph.D. Qualifying Exam

1. Let $1 \leq p \leq \infty$, $f \in L^p([0,1])$ and $\lambda(t)$ be the Lebesgue measure of the set $\{x \in [0,1] \mid |f(x)| > t\}$ for $0 \leq t < \infty$.
   (a) (5 %) Show that $h : [0, \infty) \longrightarrow [0,1]$ is measurable almost everywhere with respect to the Lebesgue measure.
   (b) (10 %) Show that $\int_0^\infty h(t)\, dt < \infty$, for $1 < p \leq \infty$.
   (c) (5 %) Is it true that $\int_0^\infty h(t)\, dt < \infty$, for $p = 1$? Justify your answer.

2. (15 %) Let $f : [0,1] \longrightarrow \mathbb{R}$ be a Lebesgue measurable function such that $f \cdot g \in L^1([0,1])$ for all $g \in L^2([0,1])$. Is it true that $f \in L^2([0,1])$? Justify your answer.

3. Let $f : [0,1] \longrightarrow \mathbb{R}$ be continuous.
   (a) (7 %) Show that $\lim_{k\to\infty} \int_0^1 x^k f(x)\, dx = 0$.
   (b) (8 %) Compute $\lim_{k\to\infty} k \int_0^1 x^k f(x)\, dx$, if possible; otherwise explain why.

4. (10 %) Let g be an integrable function and $\{f_n\}$ be a sequence of integrable functions such that $|f_n| \leq g$ a.e. for all n. Show that if $f_n \to f$ in measure μ then f is an integrable function and $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$.

5. (10 %) Let (X, Σ, μ) be a measure space and f be an integrable function. Show that for every $\epsilon > 0$ there is $E \in \Sigma$ such that $\mu(E) < +\infty$ and $\int_{X\setminus E} |f| < \epsilon$.

6. (15 %) Let f be a function defined and bounded in $Q = \{(x,t) | 0 \leq x \leq 1,\ 0 \leq t \leq 1\}$. Suppose that
   (1) $f(\cdot, t)$ is a measurable function of x for each t.
   (2) the partial derivative $\frac{\partial f}{\partial t}(x,t)$ exists for each $(x,t) \in Q$
   (3) $\frac{\partial f}{\partial t}(x,t)$ is bounded in Q.
   Show that
$$\frac{d}{dt}\int_0^1 f(x,t)dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)dx$$

7. (15 %) Let f be a integrable function in $(-\infty, \infty)$. Evaluate
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x-n)\left(\frac{x}{1+|x|}\right) dx$$