交通大學應用數學系博士班資格考(2010 年 9 月)

PDE Qualifying Exam.

September, 2010

1. (20 pts) Assume (u, v) is a C^1 complex-valued solution of the following system

$$\begin{cases} i\,u_t + |v|^2(u^* u_x + u u_x^*) = 0 & \text{in } \mathbb{R} \times (0, T)\,, \\ i\,v_t + |u|^2(v^* v_x + v v_x^*) = 0 & \text{in } \mathbb{R} \times (0, T)\,, \\ (u, v) = (g, h) & \text{on } \mathbb{R} \times \{t = 0\}\,, \end{cases}$$

where $i = \sqrt{-1}$, $*$ is the complex conjugate, T is a positive constant, g and h are smooth functions with compact support.

(i) Is there any conservation law for this system of equations? (10 pts)

(ii) Can the system have finite-time blow-up behavior for some $T > 0$? (10 pts)

Prove or disprove your answers.

2. (20 pts) Let u solve the wave equation

$$\begin{cases} u_{tt} - \Delta\, u = 0 & \text{for } x \in \mathbb{R}^3\,, t > 0\,, \\ u = g(x) & \text{for } x \in \mathbb{R}^3\,, t = 0\,, \\ u_t = h(x) & \text{for } x \in \mathbb{R}^3\,, t = 0\,, \end{cases}$$

where g, h are smooth and have compact support. Can there exist a constant C such that

$$|u(x,t)| \le C/t \quad \text{for } x \in \mathbb{R}^3\,, t > 0\,?$$

Prove or disprove all your answers.

3. (20 pts) Determine the following equations which may NOT have regularity theorem: (A) heat equation, (B) wave equation, and (C) Laplace equation. Prove or disprove your answer.

4. (20 pts) Consider the following PDE (partial differential equation)

$$\Delta\psi = e^{\psi} - e^{-\psi} \quad \text{in} \quad \mathbb{R}^n\,.$$

(i) Must there exist a positive solution ψ of the PDE such that $\sup_{x \in \mathbb{R}^n} \psi(x) < \infty$? (10 pts)

(ii) Let ψ be a smooth solution of the PDE. Must ψ be analytic in $\mathbb{R}^n$ i.e. ψ is analytic at every point of $\mathbb{R}^n$? (10 pts)

Prove or disprove your answer.

5. (20 pts) Consider the following problem

$$\begin{cases} u_t - \Delta\, u + cu = 0 & \text{for } x \in \mathbb{R}^2\,, t > 0\,, \\ u = g(x) & \text{for } x \in \mathbb{R}^2\,, t = 0\,, \end{cases}$$

where c is a positive constant and g is a smooth function with compact support. Must the solution u decay to zero as time t goes to infinity i.e. $\lim_{t\to\infty} u(x,t) = 0$ for $x \in \mathbb{R}^2$? Prove or disprove your answer.