交通大學應用數學系博士班資格考(2010 年 9 月)

幾何

1 (25 pts). Let S be a (regular, oriented) surface in Euclidean 3-space R^3 and $\mathbf{x} : U \subset R^2 \to S$ be a local parametrization in the orientation of S, giving the first fundamental form of S: $Edu^2 + 2Fdudv + Gdv^2$. a) (15 pts) Describe and compute the Christoffel symbols in terms of E, F, G and their derivatives. b) (10 pts) Let $\mathbf{x}(u(t), v(t)) = \alpha(t)$ be the expression of the (differentiable) curve $\alpha(t) \subset S$, and let w be a differentiable vector field in $\mathbf{x}(U)$ such that $w|_{\alpha(t)} = a(u(t), v(t))\mathbf{x}_u + b(u(t), v(t))\mathbf{x}_v$, also written in short by $a(t)\mathbf{x}_u + b(t)\mathbf{x}_v$ (where $\mathbf{x}_u$ denotes the usual derivative with respect to u). Compute the covariant derivative $\frac{Dw}{dt}$, and find the conditions for w to be parallel along α.

2 (25 pts). Let S be an orientable surface in R^3, and R a simple region of S with piecewise smooth boundary. a) (10 pts) State (without proof) the (local) Gauss-Bonnet Theorem for R together with proper definitions of mathematical terms contained in the statement of the theorem. b) (15 pts) Let $K(p)$ denote the Gaussian curvature of S at $p \in S$. Suppose that $|K(p)| \leq 1$ for all $p \in S$, also that the total area of S is strictly less than π. Prove that the boundary of R cannot be formed by two geodesics intersecting orthogonally both at one point P_1 from which they start and at the other point $P_2 \neq P_1$ which they meet again.

3 (25 pts). Let X, $\tilde{X}$ be smooth manifolds, and $p : \tilde{X} \to X$ be continuous. Given $x \in X$ and $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x$, let $\pi_1(X, x)$ (resp. $\pi_1(\tilde{X}, \tilde{x})$) denote the fundamental group of X (resp. $\tilde{X}$). a) (10 pts) State (without proof) the conditions for the pair $(\tilde{X}, p)$ to be a covering space of X. If $(\tilde{X}, p)$ is a covering space of X, describe (without proof) the set $p^{-1}(\{x\})$ in terms of $\pi_1(X, x)$, $\pi_1(\tilde{X}, \tilde{x})$ and p above. b) (15 pts) Assume $\tilde{X} = X = S^2$ the two-sphere, and that p is differentiable such that $dp_{\tilde{x}} : T\tilde{X}_{\tilde{x}} \to TX_x$ between tangent spaces is an isomorphism for each $\tilde{x} \in \tilde{X}$. Prove that $p : S^2 \to S^2$ is actually a diffeomorphism.

4 (25 pts). Let M be a smoothly triangulated manifold of dimension m. a) (15 pts) Define (without proof) the De Rham cohomology group of M. Then describe (without proof) the natural map from each De Rham cohomology group to the corresponding simplicial cohomology group of M in two steps: first define the map at the level of cochains, then descends the map to cohomology classes by Stoke's theorem. Finally state (without proof) the De Rham's theorem. b) (10 pts) Let $m = 3$ and assume M is simply connected. Let ω be any (non-zero) closed 2-form on M. Prove that ω must be exact. (You may use the fact that M's Euler characteristic $\chi = 0$ if needed.)