Differential Equations Qualifying Exam.

September, 2010

\# 1. (20 pts) Consider the IVP (initial value problem)

$$\begin{cases} u'' = e^u & \text{for} \quad 0 < t < T_0\,, \\ u(0) = u'(0) = 1\,, \end{cases}$$

where $u = u(t)$ is a real-valued function and T_0 is a positive constant. Must the solution u blowup at finite time i.e. there exists $t_0 > 0$ such that $\lim\limits_{t \to t_0-} u(t) = \infty$? Prove or disprove your answer.

\# 2. (20 pts) Let $(v, w) = (v(t), w(t))$ be the positive (i.e. $v(t), w(t) > 0$ for $t > 0$) solution of the IVP

$$\begin{cases} v'' = w^2\, v & \text{for} \quad t > 0\,, \\ w'' = v^2\, w & \text{for} \quad t > 0\,, \\ v(0) = w(0) = 1\,, \\ v'(0) = -w'(0) = a\,, \end{cases}$$

where a is a positive constant.

(i) Must $\lim\limits_{t \to \infty} v(t)\, w(t) = 0$? (10 pts)

(ii) Can there exist $b > 0$ such that $v(b) = w(b)$? (10 pts)

Prove or disprove your answers.

\# 3. (20 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero smooth function such that $x \cdot f(x) \leq 0$ for $x \in \mathbb{R}^2$. Let $\phi = \phi(t) \in \mathbb{R}^2$ be the solution of ordinary differential equation $\phi' = f(\phi)$ for $t > 0$ with initial data $\phi(0) = \phi_0 \in \mathbb{R}^2$.

(i) Must $\sup\limits_{t>0} \|\phi(t)\| < \infty$? Here $\|\cdot\|$ is the standard two norm of $\mathbb{R}^2$. (10 pts)

(iii) Suppose $f(0) = 0$. Must the equilibrium $\phi \equiv 0$ be stable? (10 pts)

Prove or disprove all your answers.

\# 4. (20 pts) Consider the linear system of ordinary differential equations as follows:

$$\begin{cases} \dot{x} = A\, x & \text{for} \quad t > 0\,, \\ x(0) = e\,, \end{cases}$$

where the solution $x = x(t) \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ and $A = (a_{ij})$ is a $n \times n$ matrix satisfying

$$a_{ij} = \frac{i}{j} \quad \text{for } i, j \in \{1, \cdots, n\}.$$

(i) Can you find a nonzero vector $e \in \mathbb{R}^n$ such that the solution satisfies

$$\lim_{t \to \infty} \|x(t)\| = 0 \,?$$

(10 pts)

(ii) Can you find a nonzero vector $e \in \mathbb{R}^n$ such that the solution is periodic? (10 pts)

Here $\| \cdot \|$ is the standard two norm of $\mathbb{R}^n$. Prove or disprove all your answers.

5. (20 pts) Consider the linear system of ordinary differential equations with time-dependent coefficients as follows:

$$\begin{cases} \dot{x} = B(t)\, x & \text{for } t > 0, \\ x(0) = e, \end{cases}$$

where the solution $x = x(t) \in \mathbb{R}^n$, $e \in \mathbb{R}^n$ and $B : \mathbb{R} \to \mathbb{R}^n$ is a smooth and periodic function with the period $T > 0$. Must the solution x also be periodic with the period $T > 0$? Prove or disprove your answer.