交通大學應用數學系博士班資格考(2010 年 9 月)

DEPARTMENT OF MATHEMATICS
CHIAO TUNG UNIVERSITY
Ph. D. Qualifying Examination
September, 2010
Analysis
(TOTAL 100 PTS)

Throughout this exam, $|\cdot|$ and dx denote the Lebesgue measure, χ_A denotes the characteristic function on A, and $\lambda_f(\alpha) = |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$.

1. (50%) Prove or disprove the following statements:

   (a) Let $f : [0, \infty) \mapsto \mathbb{R}$ be Lebesgue integrable. Then f is bounded on $[0, \infty)$.

   (b) Let $f : [0, 1] \mapsto \mathbb{R}$ be continuous. If $f'(x) = 0$ a.e. on $[0, 1]$, then f is constant on $[0, 1]$.

   (c) Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Then $\lim_{\alpha \to \infty} \alpha^p \lambda_f(\alpha) = 0$.

   (d) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ with $|f_n(x)| \leq \dfrac{1}{n(\ln(n+1))^2}$ for all $n \geq 1$ and all $x \in [-\pi, \pi]$. Then $f \in L^2[-\pi, \pi]$.

   (e) Let $f_1 \leq f_2 \leq f_3 \leq \cdots$ on X and $f_n \in L^1(X, \mathcal{B}, \mu)$ for all n. Then $\displaystyle\int_X \lim_{n\to\infty} f_n d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu$.

2.(10%) Let $0 < p < 1$. Assume that $a_k \geq 0$ and $x_k \geq 0$ for all k. Prove that

$$\sum_{k=1}^{\infty} a_k x_k^p \leq \left(\sum_{k=1}^{\infty} a_k\right)^{1-p} \left(\sum_{k=1}^{\infty} a_k x_k\right)^p.$$

3.(10%) Let $f : \mathbb{R}^n \mapsto (0, \infty)$ be Lebesgue measurable. If $\lambda_f(\alpha) \leq \min\{1, 1/\alpha^2\}$ for all $\alpha > 0$, prove that $\int_{\mathbb{R}^n} |f(x)| dx \leq 2$.

4.(10%) Suppose $\phi : \mathbb{R} \mapsto \mathbb{R}$ is such that

$$\phi\left(\int_0^1 f(t)dt\right) \leq \int_0^1 \phi(f(t))dt$$

for every real bounded measurable f. Prove that ϕ is convex.

5. (10%) Let $f_n, f \in L^2[-\pi, \pi]$. Suppose that

$$\int_{-\pi}^{\pi} f_n(t)g(t)dt \longrightarrow \int_{-\pi}^{\pi} f(t)g(t)dt \qquad (\text{as } n \to \infty)$$

for all $g \in L^2[-\pi, \pi]$. Is $f_n \to f$ in L^2-norm? Give your reason.

6. (10%) Let $T \in (\ell^2)^*$ with $T(e_n) = 0$ for all $n \geq 1$, where e_n is the sequence with 1 at the nth place and 0 otherwise. Prove that there is some constant α such that $T(\{a_n\}_{n=0}^{\infty}) = \alpha a_0$ for all $\{a_n\}_{n=0}^{\infty} \in \ell^2$.