Ph.D. Qualifying Examination
Numerical Analysis
Spring, 2010 <博士班資格考>

Please write down all the detail of your computation and proof.

1. (15%) Let γ be a simple root of smooth nonlinear equation $f(x) = 0$. Apply the Newton method to find γ. Show that this Newton iteration converges quadratically if the initial guess is sufficiently close to γ. What happens if γ is a double root?

2. (15%) Let T be an $n \times n$ matrix and $\mathbf{v}$ be an n dimensional column vector. Prove that the iterative method $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{v}$ converges if, and only if, the spectral radius $\rho(T) < 1$. Find the limit of this iteration if it converges. Please provide the detail for all theorems you use in the proof.

3. (15%) Given smooth function $f(x)$ and x_0, x_1, x_2, where $x_0 \neq x_2$. Show that there is a unique cubic polynomial $p(x)$ such that
$$p(x_0) = f(x_0),\ p'(x_1) = f'(x_1),\ p''(x_1) = f''(x_1),\ p(x_2) = f(x_2).$$
Derive a formula for $p(x)$.

4. (15%) Derive the trapezoidal rule and composite trapezoidal rule for numerical integration with error formula.

5. (15%) (1) Derive the Euler's method to solve the initial value problem of ODEs
$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a, b] \\ y(a) = y_0 \end{cases}$$
with local truncation error.

   (2) Apply the Euler's method to solve
$$\begin{cases} y'(t) = \sqrt{y(t)}, & t \in [0, 1] \\ y(0) = 0 \end{cases}$$
and compare the numerical solution to the exact solution. What goes wrong?

6. (15%) (1) Give a finite difference method of uniform grids for the boundary value problem
$$\begin{aligned} y'' &= p(x)y' + q(x)y + r(x) \text{ in } [a, b], \\ y(a) &= \alpha, \quad y(b) = \beta. \end{aligned}$$
What is its local truncation error?

   (2) Suppose that p, q and r are continuous and $q(x) \geq 0$ in $[a, b]$. Give a criterion, in terms of $\max_{a \leq x \leq b} |p(x)|$, for step size h that your finite difference method has a unique solution. Explain why it is true.

7. (10%) Consider the partial differential equation $u_t + u_x = 0$ with the forward-time forward-space scheme
$$\frac{v_m^{n+1} - v_m^n}{k} + \frac{v_{m+1}^n - v_m^n}{h} = 0.$$
Here $v_m^n = v(nk, mh)$ is the approximated solution of u at (nk, mh) with k and h the time and space increments, respectively. Prove or disprove that this scheme is convergent.