代數

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You may quote any standard results without proving them, but state clearly what facts you are assuming. Answers without explanation may receive no credit. Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the integers, rational numbers, real numbers, and complex numbers, respectively.

1. Let F_3 denote the field of 3 elements. Let

$$A := F_3[x]/\langle x^3 - x + 1\rangle \quad \text{and} \quad B := F_3[x]/\langle x^3 + x + 1\rangle$$

be the quotient rings where $\langle f(x)\rangle$ denotes the ideal of the polynomial ring $\mathbf{F}_3[x]$ generated by $f(x)$.

(1) [5%] Is the polynomial $x^{27} - x \in F_3[x]$ divisible by $x^3 - x + 1 \in F_3[x]$?

(2) [5%] Are rings A and B isomorphic?

(3) [5%] Describe the additive group structure of A by the fundamental theorem of finitely generated abelian groups.

(4) [5%] Let $\phi : A \to \mathbb{C}^\times$ be a nontrivial group homomorphism where $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers. Describe the image of ϕ.

2. A group G is called a *solvable group* if there exists a series of subgroups:

$$\{1\} = G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0 = G$$

(for some n) such that G_i is normal in G_{i-1} and the quotient G_{i-1}/G_i is abelian for each $i = 1, \ldots, n$. Such a series of subgroups is called a *solvable series*. A solvable group G is *polycyclic* if it has a solvable series such that G_{i-1}/G_i is cyclic.

(1) [5%] Show that a group of order 250 is solvable.

(2) [5%] Give an example of a solvable group which is not polycyclic.

(3) [10%] Prove that every homomorphism image of a polycyclic group is also polycyclic.

3. The *Jacobson radical* $J(R)$ of a ring R with unity is defined to be the intersection of all maximal left ideals of R.

(1) [5%] Find $J(\mathbb{Z})$.

(2) [5%] Suppose that $R := \{\frac{m}{n} \in \mathbb{Q} \mid n \text{ is odd}\}$. Show that

$$J(R) = \{\tfrac{m}{n} \in \mathbb{Q} \mid n \text{ is odd and } m \text{ is even}\}.$$

(3) [10%] Suppose that $f : R \to S$ is a surjective ring homomorphism. Show that $f(J(R)) \subset J(S)$.

4. Let R be a commutative ring with unity. Let A, B, C be R-modules with (R-module) homomorphisms $f : A \to C$ and $g : B \to C$. Let $A \times_C B$ denote the subset of all pairs (a, b) with $a \in A$ and $b \in B$ such that $f(a) = g(b)$. Let $p_1 : A \times_C B \to A$ be the projection $p_1(a, b) = a$ and $p_2 : A \times_C B \to B$ be the projection $p_2(a, b) = b$.

(1) [5%] Show that $A \times_C B$ is a submodule of $A \times B$.

(2) [5%] Show that $f \circ p_1 = g \circ p_2$.

(3) [5%] Suppose that there exist an R-module Z and homomorphisms $p_1' : Z \to A$, $p_2' : Z \to B$ such that $f \circ p_1' = g \circ p_2'$. Show that there exists a unique homomorphism $\phi : Z \to A \times_C B$ such that $p_1' = p_1 \circ \phi$ and $p_2' = p_2 \circ \phi$.

(4) [5%] Suppose now $B = 0$. Show that $A \times_C B$ is isomorphic to the kernel of f.

5. Let $n \geq 2$ be a positive integer, and let Ψ_n denote the set of all primitive n-th roots of unity in $\mathbb{C}$, i.e., Ψ_n is the set of generators of the group of n-th roots of unity in $\mathbb{C}$. Define

$$\Phi_n(x) := \prod_{\alpha \in \Psi_n} (x - \alpha).$$

(1) [5%] Show that $\Phi_n(x)$ is in $\mathbb{Q}[x]$ and the degree of $\Phi_n(x)$ is equal to the Euler phi-function $\varphi(n)$.

(2) [5%] Show that

$$x^n - 1 = \prod_{d \mid n} \Phi_d(x)$$

where the product is over all divisors d of n.

(3) [5%] Find $\Phi_{12}(x)$.

(4) [5%] Show that $\Phi_{2n}(x) = \Phi_n(-x)$ for all odd integers $n > 1$.